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EVOLUTIONARY GAMES ON VISIBILITY GRAPHS

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We show that time series of different complexities can be transformed into networks that host individuals playing evolutionary games. The irregularity of the time series is thereby faithfully reflected in the fraction of cooperators surviving the evolutionary process, thus effectively linking time series with evolutionary games. Pivotal to the linkage is a simple visibility algorithm that transforms time series into networks. More specifically, periodic series yield regular networks, chaotic series yield random networks, while fractal series yield scale-free networks. As an example, we use a chaotic time series from the Logistic map and a fractal time series of Brownian motion, yielding an interaction network with an exponential and a power-law degree distribution, respectively. By employing the prisoner's dilemma and the snowdrift game, we demonstrate that such heterogeneous interaction networks facilitate the evolution of cooperation if compared to the traditional square lattice topology. Due to the simplicity of the employed methodology, newcomers with a basic command of nonlinear dynamics or stochastic processes can become easily acquainted with evolutionary games, and moreover, integrate these interesting and vibrant subfields of physics more effectively into their research.

Keywords: Evolutionary games; time series; complex networks; visibility algorithm.

1. Introduction

Evolutionary games on graphs [1] offer fascinating insights into how and why cooperators can coexist with defectors in a competitive, success driven environment. One of the fundamental observations thereby has been that heterogeneous interaction networks promote cooperation in different types of social dilemmas [2]. Recent research efforts have been focused on further disentangling the role of heterogeneity by the evolution of cooperation, not just in terms of the underlying interaction network [3–9], but also in general [10–13]. In addition, following the inspiring earliest efforts [14–16], coevolutionary rules have been introduced that may generate appropriate heterogeneities spontaneously [17].

Given the existence of an impressive array of different algorithms for the generation of complex networks [18, 19], a novice can easily feel overwhelmed by which algorithm to choose and how to approach the subject. Apart from mainstream algorithms for the generation of scale-free [20] and small-world [21, 22] networks, common variations include regular small-world and regular random graphs as well as networks subject to assortative or disassortative mixing, to name a few. Here we demonstrate that the visibility algorithm [23] can yield suitable interaction networks depending on the complexity of the time series that is used as input, and more importantly, that the irregularity of the trace is then directly reflected in the outcome of evolutionary games played on the resulting graphs.

Time series analysis is a fascinating field of research [24], and there are several methods and approaches developed for quantifying the complexity of observed traces [25]. Notably, recent advances also include complex networks as a means to differentiate between periodicity, deterministic chaos and noise [26–28]. Here, however, the emphasis is not on using the outlined approach to characterize time series, but rather to use potentially existing knowledge about nonlinear dynamics and stochastic processes to bridge the gap between two seemingly very different fields of research. Using as the input a time series with a given complexity, e.g. periodic, chaotic, random or fractal [24], the visibility algorithm retrieves an interaction network of which the degree distribution mirrors the properties of the series. More precisely, chaotic and random series convert into networks with an exponential degree distribution, while fractal series yield as output networks with a power-law degree distribution. Thus, it is possible to link the existing knowledge about nonlinear dynamics and stochastic processes with the complexity of networks, and further with evolutionary games.

Subsequently, networks generated in this way can be used to demonstrate the impact of network heterogeneity on the evolution of cooperation in the light of previous results obtained on square lattices [29–32]. From the set of social dilemmas we here consider the evolutionary prisoner's dilemma and the evolutionary snowdrift game as the two most representative examples [33, 34]. We demonstrate that networks with exponential and power-law degree distributions strongly facilitate the evolution of cooperation irrespective of the governing social dilemma, thus capturing the essence of recent advances in evolutionary games on complex networks by means of a straightforward approach, simply by switching the input time series for the visibility algorithm. In the following sections, we give a description of the visibility algorithm that converts time series into networks, and subsequently present the outcome of evolutionary games in dependence on the complexity of the time series that is used as input and the temptation to defect.

2. Visibility Algorithm

Given a time series, the corresponding visibility graph is obtained by treating every point of the series $x_{i=1,\ldots,N}$ as a node (i.e. the length of the time series thus

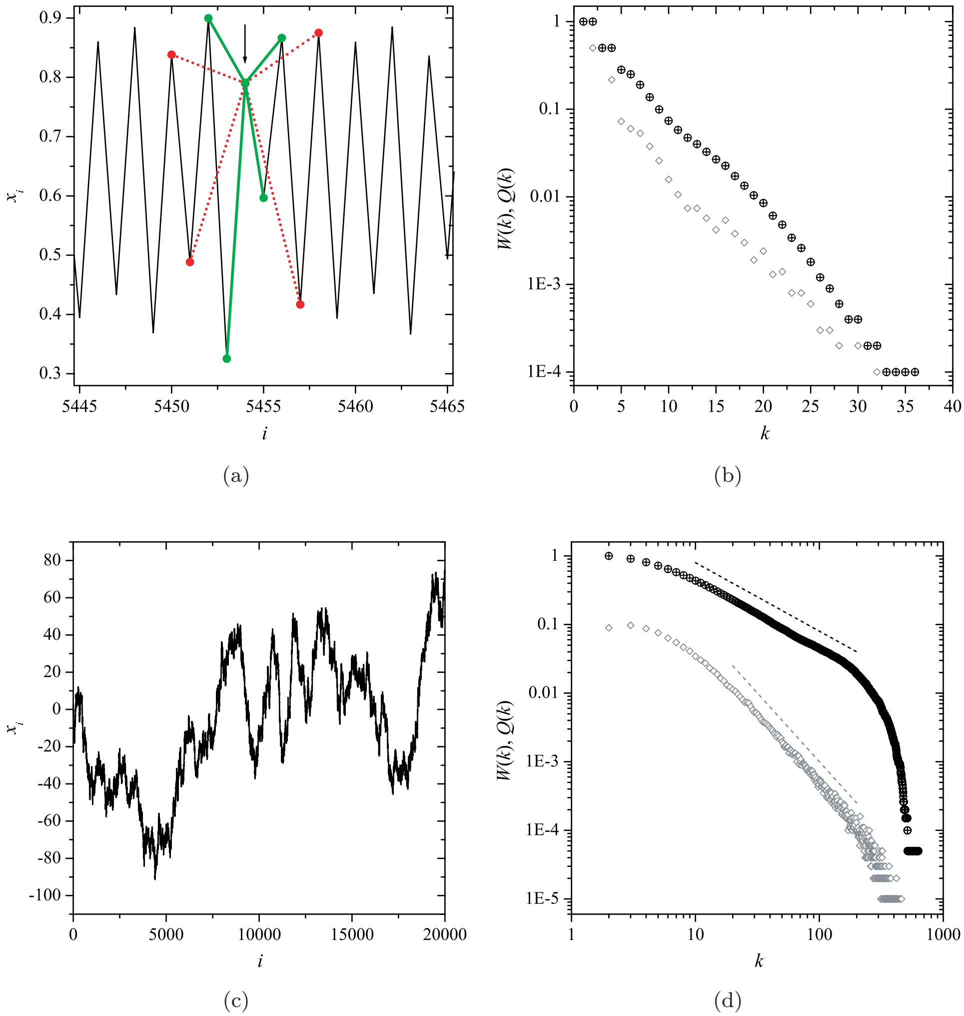

Fig. 1. (Color online) Visibility algorithm and the resulting networks. (a) Short segment (out of $N = 10^4$ points in total) of the logistic map series obtained for $r = 3.6$, featuring a schematic presentation of the visibility algorithm. The node marked by the arrow is connected to all green nodes (green solid lines) but not to the red ones (dotted red lines), since the latter do not fulfil the visibility condition given by Eq. (1). (b) Degree distribution $W(k)$ (gray $\diamond$) and the cumulative degree distribution $Q(k)$ (black $\oplus$) of the network obtained by using the chaotic time series of the Logistic map [see panel (a)] as input. Since the vertical axis has a logarithmic scale and both data sets can be fitted fairly accurately by a straight line with an identical slope the degree distribution is exponential. (c) First $N = 2 \cdot 10^4$ points of the Brownian series, corresponding to a random walk in one dimension. (d) Degree distribution $W(k)$ (gray $\diamond$) and the cumulative degree distribution $Q(k)$ (black $\oplus$) of the network obtained by using the Brownian time series [see panel (c)] as input. Since both axis have a logarithmic scale and $W(k) \propto k^{-\alpha}$ as well as $Q(k) \propto k^{-(\alpha-1)}$ the degree distribution is a power-law with $\alpha \approx 2.0$ (as indicated by the two dashed lines).

corresponds to the network size N) and connecting a given node with all those nodes that can be "seen" from the top of it [23]. Figure 1(a) features a short segment of the logistic map $x_i = rx_{i-1}(1 - x_{i-1})$ obtained at $r = 3.6$, where the algorithm is schematically presented. Green lines depict valid links while red lines depict forbidden links. Note that the nodes depicted red cannot be seen from the node marked by the arrow without intersecting the series at least once. On the other hand, all nodes marked green are directly visible to one another. Taking into consideration basic geometric relations, it is possible to derive a simple criteria for the visibility of any two data points. In particular, two arbitrary points of the series (i, x_i) and (j, x_j) will have visibility, and thus will become two connected nodes in the corresponding visibility graph, if any other point (m, x_m) placed between them fulfils [23]:

$$x_m < x_j + (x_i - x_j)\frac{j - m}{j - i}. \tag{1}$$

In order to obtain the visibility graph from a time series one has to check the condition given by Eq. (1) for every possible pair of points, whereby taking into account all the points that are placed between them. If the visibility criteria are fulfilled for all m the two points i and j should be connected, but otherwise not. This simple procedure warrants that the visibility graph is always connected since each point is connected to at least its two nearest neighbors (left and right), and moreover, is undirected since the algorithm does not distinguish between different link directions. These properties make the obtained networks suitable candidates as underlying interaction topologies for evolutionary games.

We characterize the resulting networks by means of the degree distribution and the cumulative degree distribution [19]. By defining k_i as the degree of node i, the degree distribution $W(k)$ gives the probability that a node chosen uniformly at random has degree k. As a very useful alternative, the cumulative degree distribution $Q(k)$ can be defined as the probability that a node chosen uniformly at random has degree at least k (i.e. k or smaller). Note that if $W(k) \propto k^{-\alpha}$ (is a power-law with slope α), then also $Q(k)$ will be a power-law, but with the slope $\alpha - 1$ rather than α. Thus, having $W(k) \propto k^{-\alpha}$ and $Q(k) \propto k^{-(\alpha-1)}$ is a firm indicator of a scale-free network. On the other hand, if $W(k) \propto \exp^{-k/\kappa}$ (is exponential with slope κ) then $Q(k)$ will also be exponential, but with the same exponent [19]. Thus, plotting $W(k)$ and $Q(k)$ on logarithmic or semi-logarithmic scales makes it easy to distinguish power-law from exponential distributions.

If one considers as input a periodic time series, it is straightforward to reckon that the visibility criteria will be periodically fulfilled every oscillation period. Accordingly, the resulting network will be regular having a discrete degree distribution with a finite number of peaks corresponding to the number of points forming one period of the series. Much more interesting scenarios are possible if one uses random, chaotic or fractal series as input. First, it is important to realize that any large value (larger than the surrounding values) of the time series will map to a hub of the corresponding visibility network. Second, for random as well as chaotic time series it holds that two consecutive extreme values of the series are highly improbable. In fact, the time distribution of extreme events in a sequence of uniformly distributed random numbers is exponential. From these two facts it follows directly that the networks constructed from a random or a chaotic time series will have an exponential degree distribution. Indeed, in Fig. 1(b) this line of thought is fully confirmed, whereby as input we have used a series from the well-known Logistic map $x_i = rx_{i-1}(1 - x_{i-1})$ with $r = 3.6$ (note that for this value of the parameter the map is chaotic) [35]. Extending the outlined reasoning further, one finds that time series that violate the exponentially infrequent occurrence of extreme events, such as for example those that are fractal, convert to networks having a power-law degree distribution. Figure 1(d) confirms this expectation, whereby as input we have used the random walk in one dimension, i.e. the Brownian series. From these results it follows that the described visibility algorithm offers fascinating possibilities with respect to fast, efficient and extremely versatile generation of complex networks, which can all be applied as underlying interaction topologies when studying the evolution of cooperation in the context of evolutionary games.

3. Evolutionary Games

In what follows, both the prisoner's dilemma game as well as the snowdrift game will be used as representative examples of social dilemmas, whereby we adopt the same parametrization as used recently in Ref. 7. Accordingly, the prisoner's dilemma game is characterized by the temptation to defect $T = b$, reward for mutual cooperation $R = 1$, and punishment P as well as the suckers payoff S equaling 0, whereby $1 < b \leq 2$ ensures a proper payoff ranking [29]. The snowdrift game, on the other hand, has $T = \beta$, $R = \beta - \frac{1}{2}$, $S = \beta - 1$ and $P = 0$, where the temptation to defect can be expressed in terms of the cost-to-benefit ratio $r = 1/(2\beta - 1)$ with $0 \leq r \leq 1$. In both games two cooperators facing one another acquire R, two defectors get P, whereas a cooperator receives S if facing a defector who then gains T. Initially each player i, corresponding to a node of the underlying network, is designated either as a cooperator (C) or defector (D) with equal probability. Irrespective of the game, evolution of the two strategies is performed in accordance with the Monte Carlo simulation procedure comprising the following elementary steps. First, a randomly selected player i acquires its payoff p_i by playing the game with all its k_i neighbors. Next, one randomly chosen neighbor of i, denoted by j, also acquires its payoff p_j by playing the game with all its k_j neighbors. Last, if $p_i > p_j$ player i tries to enforce its strategy s_i on player j in accordance with the probability $W(s_i \to s_j) = (p_i - p_j)/bk_q$, where k_q is the largest of the two degrees k_i and k_j. In accordance with the random sequential update, each player is selected once on average during a full Monte Carlo step. Presented results were obtained on networks hosting $N = 10^4$–10^5 players and the equilibrium fractions of cooperators ρ_C were determined within 10^6 full Monte Carlo steps after sufficiently long transients were discarded.

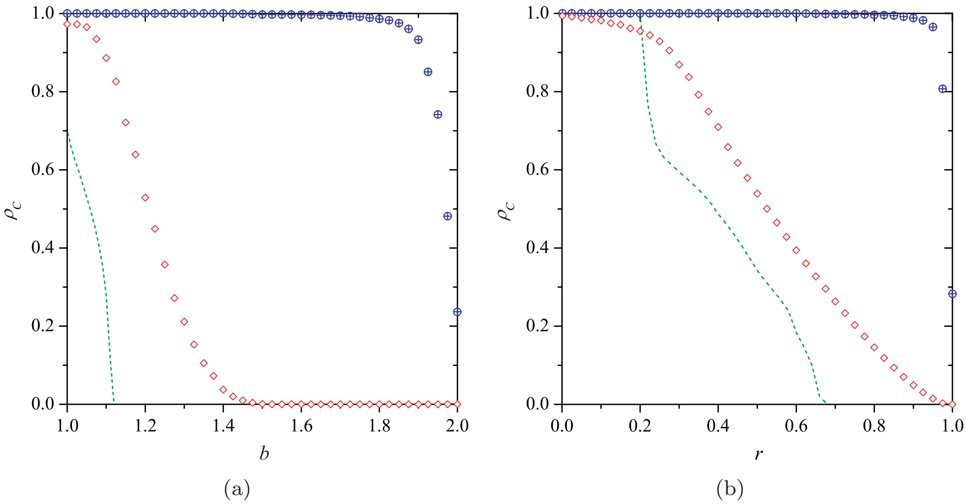

Fig. 2. (Color online) Evolution of cooperation. (a) Fraction of cooperators ρ_C in dependence on the temptation to defect b for the prisoner's dilemma game. (b) Fraction of cooperators ρ_C in dependence on the cost-to-benefit ratio r for the snowdrift game. In both panels the dashed green line depicts results obtained on the square lattice, while red $\diamond$ and blue $\oplus$ depict results obtained on the network with an exponential [see Fig. 1(b)] and a power-law [see Fig. 1(d)] degree distribution, respectively. It can be observed that heterogeneous interaction topologies in form of above-introduced visibility graphs strongly promote the evolution of cooperation irrespective of the governing social dilemma.

In Fig. 2 we present the main results of the evolutionary process for the two considered games. Taking the evolution of cooperation on the square lattice [30, 36] as a benchmark (dashed green lines in both panels of Fig. 2), it is inferable at a glance that heterogeneous interaction networks have a very positive effect on the survivability of cooperators. While cooperators on the square lattice die out at $b \approx 1.12$ and $r \approx 0.68$ in the prisoner's dilemma and the snowdrift game, respectively, they prevail across large spans of b and r if networks with an exponential (red $\diamond$ in both panels of Fig. 2) or a power-law (blue $\oplus$ in both panels of Fig. 2) degree distribution are used as underlying interaction topologies. Comparatively, it can be observed that scale-free networks are more efficient in promoting the evolution of cooperation than networks with an exponential degree distribution. This is in agreement with the results reported in several previous studies [37–39], where it was shown that the degree heterogeneity of scale-free networks, along with the interconnectedness of hubs, strongly reinforces cooperative behavior. It is also worth pointing out that the networks generated by means of the visibility algorithm yield very similar results as networks generated with the more traditional algorithms; for example the one proposed by Barabási and Albert [20] (compare with Fig. 1 in Ref. 37). Importantly, while on the square lattice cooperators form clusters to protect themselves against being exploited by defectors [29], on heterogeneous interaction networks hubs (i.e. nodes with a high degree) act as robust sources of cooperative behavior. This difference in the way cooperators defend themselves against defectors is also the main reason for the facilitative effect of heterogeneity on the evolution of cooperation, which thus cannot be observed in this form on regular lattices and graphs. The promotion of cooperation via heterogeneity has been a source of inspiration ever since its discovery [37], and herewith we shown that this fascinating result can be reproduced elegantly by means of evolutionary games on networks that are generated by means of the visibility algorithm.

4. Summary

In summary, we have outlined a simple approach that links time series with the outcome of evolutionary games, thereby enabling graduate students and teachers to become easily acquainted with different subfields of physics by means of an interdisciplinary approach. Since the visibility algorithm enables altering the network properties simply by switching the input time series, it is possible to study how different levels of heterogeneity influence the outcome of evolutionary games in an effective and accessible manner. By using networks derived from a chaotic Logistic map and the Brownian motion, we have shown that networks with exponential and power-law degree distributions facilitate the evolution of cooperation across a wide span of defection temptation values and irrespective of the governing social dilemma. Especially if compared to the outcome of games on a square lattice the facilitative impact on the evolution of cooperation is remarkable and very convincing. We hope the study will succeed in drawing further attention to this currently very vibrant field of research [1, 17, 40–44].

Acknowledgments

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