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# Extended matrix norm method: Applications to bimatrix games and convergence results

Burhaneddin İzgi [a,c,*], Murat Özkaya [a,d], Nazım Kemal Üre [b,c], Matjaž Perc [e,f,g,h]

[a] Istanbul Technical University, Department of Mathematics, Maslak, Istanbul, 34469, Turkey
[b] Istanbul Technical University, Department of Artificial Intelligence and Data Eng., Maslak, Istanbul, Turkey
[c] Artificial Intelligence and Data Science Research Center, Istanbul Technical University, Istanbul, Turkey
[d] Çanakkale Onsekiz Mart University, Faculty of Political Sciences, Business Department, Çanakkale, Turkey
[e] Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, Maribor 2000, Slovenia
[f] Complexity Science Hub Vienna, Josefstädterstraße 39, Vienna 1080, Austria
[g] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 404332, Taiwan
[h] Alma Mater Europaea, Slovenska ulica 17, Maribor 2000, Slovenia

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ABSTRACT

In this paper, we extend and apply the Matrix Norm (MN) approach to the nonzero-sum bimatrix games. We present preliminary results regarding the convergence of the MN approaches. We provide a notation for expressing nonzero-sum bimatrix games in terms of two matrix games using the idea of separation of a bimatrix game into two different matrix games. Next, we prove theorems regarding boundaries of the game value depending on only norms of the payoff matrix for each player of the nonzero-sum bimatrix game. In addition to these, we refine the boundaries of the game value for the zero/nonzero sum matrix games. Therefore, we succeed to find an improved interval for the game value, which is a crucial improvement for both nonzero and zero-sum matrix games. As a consequence, we can solve a nonzero-sum bimatrix game for each player approximately without solving any equations. Moreover, we modify the inequalities for the extrema of the strategy set for the nonzero-sum bimatrix games. Furthermore, we adapt the min-max theorem of the MN approach for the nonzero-sum bimatrix games. Finally, we consider various bimatrix game examples from the literature, including the famous battle of sexes, to demonstrate the consistency of our approaches. We also show that the repeated applications of Extended Matrix Norm (EMN) methods work well to obtain a better-estimated game value in view of the obtained convergence results.



## 1. Introduction

Game theory is a branch of mathematics that investigates optimal decision-making models in a situation of competition [1]. The theory focuses on the problems of selecting an optimal action among various types of actions [2]. Historically, game theory was first used for solving military problems. However, game theory found a wider array of applications throughout the years. For example, distribution of resources between countries, price competition between companies, or even gambling are some of the common application areas. As it is seen from these examples, the theory includes participants that interact

* Corresponding author.
*E-mail address:* bizgi@itu.edu.tr (B. İzgi).

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with each other. The subject of the game theory works on those interactions such that the choices of each participant affect the outcome that is of interest to all participants[3].

Throughout the past, researchers working on game theory were awarded Nobel Prize in economics. For example, Nash and other colleagues shared the Nobel Prize for the study of equilibria in non-cooperative games in 1994 [4]. Aumann and Schelling were awarded for enhancing the understanding of competition and cooperation through game theory [5]. The game theory progresses rapidly and interacts with every area of science over time. In 1959, Berkovitz and his colleagues investigated a tactical air war with game theory [6]. In 2008, Perc and Szolnoki demonstrated that social diversity efficiently promotes cooperation during the game of the spatial prisoner's dilemma [7]. Huttegger and Zollman studied the evolution of strategic behavior using game theoretical tools in 2013 [8]. Szolnoki and Perc presented coevolutionary success-driven multi-games in structured populations, which use the weak prisoner's dilemma as the core game in 2014 [9]. In 2016, Firouzbakth et. al presented conditions for the Nash equilibrium and they applied their game theoretical approach to derive the optimal transmission and jamming strategies for a typical wireless link under power-limited jamming [10]. Roy and Mula combined the rough programming approach and the matrix games, by developing a new class called a rough matrix game in [11]. In 2018, Bigdeli et. al solved constrained bimatrix games in the fuzzy environment and they also applied such games to nuclear negotiations between two countries [12]. In the same year, Perera investigated how a government can manage a policy of environmental sustainability in a competitive electricity market with the bimatrix coordination games [13]. In 2018, Deng et al. investigated a special class of multi-person games, that is, each connection point of the interaction graph defines a bimatrix game [14]. Cressman and Krivan, in 2020, developed a new approach to the theory of two-player asymmetric evolutionary games using the game Battle of Sexes. They showed that the evolutionary outcomes depend on the total population [15]. In 2021, Lensberg and Sheck-Hoppe studied one-shot games in the set of all bimatrix games with a large population of agents. They proposed an approach that predicts the behavior of experienced agents during the game [16]. All these examples are related to the bimatrix games, and many more examples can be presented. The rate of progress in game theory shows that the theory will continue to attract researchers.

In the literature, the nonzero-sum bimatrix games are solved by using theorems or separating the game into two different matrix games [17,18]. In light of the latter fact, most methods in the literature use linear programming or other methods to obtain solutions to matrix games. The small-scale matrix games may be solved easily with the approaches found in the literature. However, the solution procedure gets more tedious for obtaining a solution for large-scale matrix games due to the increasing number of inequalities and the computational cost. Thereupon, İzgi and Özkaya focused on solving and generating zero-sum matrix games by using norms of the payoff matrix in 2018 [19]. In particular, İzgi and Özkaya introduced the Matrix Norm approach that approximately solves the zero-sum matrix game without the need of solving any equations. According to the literature research, İzgi and Özkaya brought the game theory and the matrix norms together for the first time with their paper [19]. İzgi and Özkaya additionally use this approach for the demonstration of the fairness of a matrix game in [20]. Further, Özkaya investigated the Matrix Norm approach and the use of matrix norm in the game theory in detail [21]. In addition, İzgi and Özkaya apply the Matrix Norm approach to show the necessity of agricultural insurance and show the usefulness of the method [22] in 2020.

In this study, we extend and apply the Matrix Norm approach in [19] to the nonzero-sum bimatrix games for the first time in the literature. We especially focus on the game solution process of the Matrix Norm approach and impose the process for the bimatrix games. In order to apply the Matrix Norm method for the bimatrix case, we borrow the idea presented in [17,18], which is that we can split the bimatrix game into two different matrix games and investigate them separately, by presenting notations for the bimatrix games. Then, we first adapt and prove theorems that provide the boundaries for the game value of corresponding players by using the same proof approach used for the matrix games with the new notation of the bimatrix games. We also demonstrate the boundaries for the extrema of the strategy set of the players for the nonzero-sum bimatrix games. Moreover, we modify the min-max theorem in [19] for the nonzero-sum bimatrix games. Additionally, we advance the boundaries for the game value in [19]. Hence, we succeed to find an improved interval, which is a crucial improvement for both nonzero and zero-sum matrix games, for the game value. In addition to these, we consider the convergence of the MN method, and present some basic results regarding the convergence of the Matrix Norm approaches.

The remainder of the paper is organized as follows. In Section 2, we first present a new notation for the nonzero-sum bimatrix games. Then, we express and prove the generalized form of some theorems about the game value boundaries. Moreover, we state inequalities of upper/lower bounds of the minimum and the maximum elements of the strategy set, respectively. Furthermore, we represent the min-max theorem, which demonstrates the relationship between the largest and the smallest components of the strategy set. Additionally, we provide new theorems for boundary refinement of the game value of both zero and nonzero-sum games. In Section 3, we state a theorem about the convergence of the Matrix Norm approach and prove it. Moreover, we present a corollary that guarantees the existence of the solution and demonstrates the proof. In Section 4, we show the consistency of our approaches by using famous problems of the game theory. The final section concludes the paper and presents a discussion on the subject.

## 2. The modification of the matrix norm approach to bimatrix games

In this section, we extend the new matrix norm approach in [19] to the nonzero-sum bimatrix games. First of all, we present a new notation for the nonzero-sum bimatrix games to adapt the theorems and proofs in [19]. We introduce and comprehensively demonstrate the proofs of theorems that include the inequalities about the game value of the nonzero-sum bimatrix game. Moreover, we represent the theorem for $p^k_{\max}$ and $p^k_{\min}$ that are respectively the greatest and the smallest components of the mixed strategy sets of each player, with the new notation. We also obtain the new boundaries for the game value. Furthermore, we adapt the min-max theorem of the zero-sum matrix games, which shows the relationship between $p^k_{\max}$ and $p^k_{\min}$, to the nonzero-sum bimatrix games.

**Definition 2.1.** A game is called nonzero-sum game if the sum of players' payoffs (game values) does not equal to zero.

**Definition 2.2.** Bimatrix game is a two player normal form game where

- player 1 has a finite strategy set $S = \{s_1, s_2, \dots, s_m\}$
- player 2 has a finite strategy set $T = \{t_1, t_2, \dots, t_n\}$
- when the pair of strategies $(s_i, t_j)$ is chosen, the payoff to the first player is $a_{ij} = u_1(s_i, t_j)$ and the payoff to the second player is $b_{ij} = u_2(s_i, t_j)$ ; $u_1$, $u_2$ are called payoff functions.

The values of payoff functions are defined with a bimatrix as follows:

**Notation.** Let $A$ be real valued $m \times n$ bimatrix nonzero-sum game as in the following form

$$A = \begin{bmatrix} (a_{11}, b_{11}) & . & (a_{1n}, b_{1n}) \\ . & . & . \\ (a_{n1}, b_{n1}) & . & (a_{mn}, b_{mn}) \end{bmatrix}$$

where the pair $(a_{ij}, b_{ij})$ represents the payoff of the Player I and Player II, respectively. We denote $a_{ij}$ as $A[(i, j), 1]$ and $b_{ij}$ as $A[(i, j), 2]$. We know that we can consider this bimatrix as two different matrices as

$A_1 = \begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{n1} & . & a_{mn} \end{bmatrix}$, $A_2 = \begin{bmatrix} b_{11} & . & b_{1n} \\ . & . & . \\ b_{n1} & . & b_{mn} \end{bmatrix}$ where $A_1 = A[(i, j), 1]$ and $A_2 = A[(i, j), 2]$ are the payoff matrices for the first and second players, respectively. Contrary to the zero-sum matrix game, we have two different game values in this case as $v_1$ and $v_2$ whose total sum is not a constant, that is, $v_1 + v_2 \neq c$ where $c$ is constant. This means that knowing the game value of any player does not give any idea about the other's game value. It is the main reason these types of games are called nonzero-sum games [17]. Therefore, we need to solve two different matrix games. On the other hand, if $A_1 = -A_2$, then the game is a zero-sum matrix game in the common sense [18]. In this situation, one may approximately solve the game directly by using the methodology presented in [19].

**Definition 2.3.** Let $A$ be real valued $m \times n$ bimatrix of the nonzero-sum game and $v_k$ be the game value of the $k$th player for $k = 1, 2$. Then, the game value is obtained by $v_k = \sum_{i=1}^{m} A[(i, j), k] p^k_i$, for any $j$ and fixed $k$, where $p^k_i$'s are the elements of the mixed strategy set of the corresponding player and $\sum_{i=1}^{m} p^k_i = 1$.

**Lemma 2.4.** *Let $A$ be real valued $m \times n$ bimatrix of the nonzero-sum game as in the following form*

$$A = \begin{bmatrix} (a_{11}, b_{11}) & . & (a_{1n}, b_{1n}) \\ . & . & . \\ (a_{m1}, b_{m1}) & . & (a_{mn}, b_{mn}) \end{bmatrix}$$

*and the game value of the kth player is $v_k$ where $k = 1, 2$. Then,*

*if $v_k \geq 0$, then $\frac{z}{||A_k||_\infty} \leq v_k \leq ||A_k||_1$*

*and*

*if $v_k \leq 0$, then $-||A_k||_1 \leq v_k \leq \frac{z}{||A_k||_\infty}$*

*hold where $z = v||B_k||_\infty$ for fixed $k$ and $B_k$ is the row-wise induced matrix, stated in Definition 2.4 in [19], of the bimatrix $A$ for the related player.*

**Proof.** We first split the bimatrix game into two different matrix games, as it is presented in [17,18], with our notation above, say $A_1$ and $A_2$. Then, we can prove the lemma directly by following the same motivation that is used for the zero-sum matrix games in [19]. □

**Corollary 2.4.1.** *Let $A$ be real valued $m \times n$ bimatrix of the nonzero-sum game, $v_k$ is the game value of the related player and $B_k$ that is the row-wise induced matrix of the bimatrix $A$ for the corresponding player. Then,*

*if $v_k \geq 1$ then $\frac{||B_k||_\infty}{||A_k||_\infty} \leq v_k \leq ||A_k||_1$,*

*and*

*if $v_k \leq -1$ then $-||A_k||_1 \leq v_k \leq -\frac{||B_k||_\infty}{||A_k||_\infty}$,*

*hold where $k = 1, 2$ and $B$ is the row-wise induced matrix of $A$.*

**Theorem 2.5.** *Assume that $A$ be real valued $m \times n$ bimatrix of the nonzero-sum game as in the following form*

$$A = \begin{bmatrix} (a_{11}, b_{11}) & . & (a_{1n}, b_{1n}) \\ . & . & . \\ (a_{m1}, b_{m1}) & . & (a_{mn}, b_{mn}) \end{bmatrix}$$

*and the game value of the kth player is $v_k$. Then,*
*if $|v_k| \geq 1$ then $\frac{||B_k||_\infty}{||A_k||_\infty} \leq |v_k| \leq ||A_k||_1$,*
*and*
*if $|v_k| \leq 1$ and $v_k \neq 0$ then $\frac{1}{||A_k||_1} \leq |v_k| \leq \frac{||A_k||_\infty}{||B_k||_\infty}$,*
*hold where for $k = 1, 2$, and $B_k$ is the row-wise induced matrix of $A_k$.*

**Proof.** We know $\frac{||B_k||_\infty}{||A_k||_\infty} \leq 1$ where $||A_k||_\infty = \max_i \sum_{j=1}^{n} |A[(i, j), k]|$ and $B_k$ is the row-wise induced matrix of $A_k$, for any $k$. We also have $v_k \leq ||A_k||_1$ when $v_k$ is positive and $-||A_k||_1 \leq v_k$ when $v_k$ is negative by Lemma 2.4.

**(i)** When $v_k$ is positive,

**(a)** If $v_k \geq 1$, then we have $\frac{||B_k||_\infty}{||A_k||_\infty} \leq v_k \leq ||A_k||_1$ by Corollary 2.4.1

**(b)** If $0 \leq v_k \leq 1$, then we suppose $t^{-1} = v_k$. Thus, we have, $t = \frac{1}{v_k} \geq 1$. Then, it follows $\frac{1}{||A_k||_1} \leq v_k \leq \frac{||A_k||_\infty}{||B_k||_\infty}$ by using the result in (a).

**(ii)** When $v_k$ is negative,

**(a)** If $v_k \leq -1$, then $v_k \leq -1 \leq -\frac{||B_k||_\infty}{||A_k||_\infty}$. We reach $-||A_k||_1 \leq v_k \leq -\frac{||B_k||_\infty}{||A_k||_\infty}$ by using Corollary 2.4.1.

**(b)** If $-1 \leq v_k \leq 0$ then we assume $s^{-1} = v_k$. Hence, $s = \frac{1}{v_k} \leq -1$. We get $-||A_k||_1 \leq s \leq -\frac{||B_k||_\infty}{||A_k||_\infty}$ by the result of (a). Then we can rewrite it as, $-\frac{||A_k||_\infty}{||B_k||_\infty} \leq v_k \leq -\frac{1}{||A_k||_1}$

After the necessary arrangements for the inequalities in (i) and (ii) the result follows. □

**Proposition 2.6.** *Any bimatrix game can be perturbed while the game value $v_k$ changes as much as the perturbation and the strategy sets remain the same.*

**Proof.** One may split the bimatrix game into two different matrix games. As a result of this, we can prove it similar to Proposition 2.7 of [19] by using the new notations. □

**Corollary 2.6.1.** *Any bimatrix game could be reduced to a game with game value $v_k = 0$ with the same strategy set.*

Note that we can use the above corollary in order to create a nonzero-sum bimatrix game whose game value is zero for any player.

**Theorem 2.7.** *Let A be real valued $m \times n$ bimatrix, where $a_{ij} > 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, of a nonzero-sum game. Then the boundaries for the greatest and the smallest elements, $p^k_{\max}$ and $p^k_{\min}$ respectively, of the strategy set of the corresponding players are as follows*

$$p^k_{\max} \geq L_k \text{ where } L_k = \max\left\{\frac{1-\frac{v_k}{||A_k||_1}}{m-1}, \frac{v_k}{||B_k||_1}\right\},$$

$$p^k_{\min} \leq U_k \text{ where } U_k = \min\left\{\frac{1-\frac{v_k}{||B_k||_1}}{m-1}, \frac{v_k}{||A_k||_1}\right\}$$

*where $B_k$ is the column wise induced matrix of $A_k$ and $k = 1, 2$.*

**Proof.** The proof can be done as it is proved for Theorem 3.3 in [19]. □

The next theorem shows the connection between the extremum values of the strategy set. The theorem also has an essential role in the game creation process.

**Theorem 2.8.** ***(Min-Max Theorem)*** *Let A be real valued $m \times n$ bimatrix of a nonzero-sum game. Then, $\frac{1-p^k_{\min}}{m-1} \leq p^k_{\max} \leq 1 - (m-1)p^k_{\min}$ and $1 - (m-1)p^k_{\max} \leq p^k_{\min} \leq \frac{1-p^k_{\max}}{m-1}$ hold where $k = 1, 2$.*

**Proof.** The proof is very similar to that of Theorem 3.2, which is the min-max theorem, in [19] and hence the details are skipped here. □

### *2.1. Extended matrix norm approach: Boundaries refinement for game value*

The following theorems improve the boundaries for the game value of zero-sum matrix games obtained by the MN method in [19].

**Theorem 2.9.** *Let A be a $m \times n$ payoff matrix and $v$ be the game value for a two person zero sum game, and define $||A||_{\bar{1}} = \min_j \sum_i |a_{ij}|$, $||A||_{\bar{\infty}} = \min_i \sum_j |a_{ij}|$. Then,*
*if $|v| \geq 1$, then $L_v \leq |v| \leq U_v$,*
*and*
*if $|v| \leq 1$, and $|v| \neq 0$, then $(U_v)^{-1} \leq |v| \leq (L_v)^{-1}$*

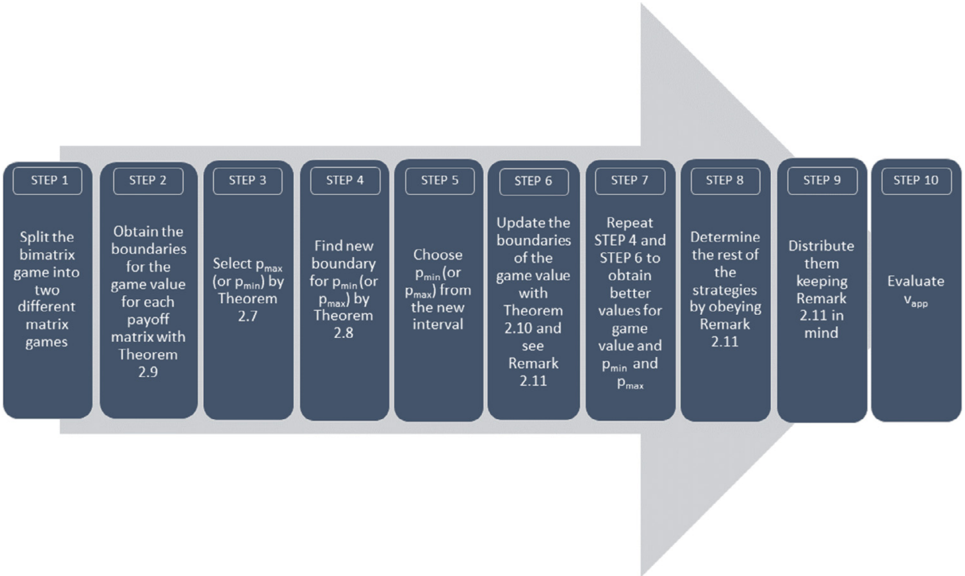


**Fig. 1.** Solution steps' diagram for the two-person nonzero-sum bimatrix games.

*while* $L_v = \max\left\{\frac{||B||_\infty}{||A||_\infty}, \frac{||B||_1}{||A||_1}\right\}$ *and* $U_v = \min\left\{||A||_{\bar{1}}, ||A||_{\bar{\infty}}\right\}$*, and B is the row-wise induced matrix of A.*

**Proof.** For $|v| \geq 1$: The demonstration for the left-hand side ($L_v$) of the inequality can be done directly using the fact that of a zero-sum game by taking into account the Theorem 2.5 in [19] in the view of Player I and Player II, separately. Similarly, one can show that the right side of this inequality holds by considering the basic probability assumption that is the strategy set elements are less than or equal to 1 (i.e. $p_i \leq 1$ for all $1 \leq i \leq n$) since $v = \sum_{i=1}^{m} p_i a_{ij}$. Correspondingly, the proof can be done by following up the same steps above for $|v| \leq 1$, as well. □

**Theorem 2.10.** *Let A be real valued* $m \times n$ *payoff matrix, where* $a_{ij} > 0$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*, of a two person zero sum game. Then,* $p_{\min}||A||_1 \leq v \leq p_{\max}||A||_{\bar{1}}$ *holds while* $||A||_{\bar{1}} = \min_j \sum_i |a_{ij}|$*, and* $p_{\min}$ *and* $p_{\max}$ *are the smallest and greatest components in the mixed strategy set, respectively.*

**Proof.** Let $||A||_{\bar{1}} = \sum_{i=1}^{m} |a_{ij}|$ for fixed $1 \leq j_0 \leq n$. We know $v = \sum_{i=1}^{m} p_i a_{ij}$ for any $j = 1, \ldots, n$ and $p_i \in [0, 1]$. We can also write $v = \sum_{i=1}^{m} p_i a_{ij} = \sum_{i=1}^{m} p_i |a_{ij_0}| \leq \sum_{i=1}^{m} p_{\max} |a_{ij_0}|$ by using these facts. Then, the result follows. The rest of the proof can be done similarly. □

It is clear that Theorem 2.10 can be also applied to nonzero-sum bimatrix games using the similarity with the zero-sum matrix games while solving the problem.

**Remark 2.11.** In order to solve a bimatrix game, after splitting the bimatrix game into two different matrix games, it is crucial to realize that the $v_{app}$ should stay in the optimized interval obtained from Theorem 2.9 and Theorem 2.10. Moreover, the strategy set's elements must be selected in the consideration of Theorem 2.7 and principles of the probability and game theories. Then, one may distribute the strategies in any order by obeying these rules.

**Remark 2.12.** The game creation process for bimatrix games can be completed by using the methodology, which is described in [19], for zero-sum matrix games.

Finally, we summarize our extended approach for the solution of bimatrix games in Fig. 1 which might be a practical tool for the players.

## 3. Convergence results of the MN approaches

The convergence of the iterative numerical methods is essential. In the illustrations part of this study, we suggest and show the use of the Extended Matrix Norm approach compared with the Matrix Norm approach iteratively works well to obtain a better interval for the game value of a matrix game. Therefore, in this section, we aim to discuss some theoretical preliminary results about the convergence of the MN method (and also the EMN method).

**Theorem 3.1.** *Let $v_{app}$ be approximated game value obtained by MN approaches while $P_{est}^{j} \in R^{1\times m}$, $j = 1, 2, \dots, m!$ presents the all possible strategy sets of zero sum matrix game with positive entries $A \in \mathbb{R}^{m\times n}$ payoff matrix such that $P_{est}^{j} a_{ij_0} = v_{app}$, for all $1 \le i \le m$ and $1 \le j_0 \le n$. Then*

$$\lim_{||P||\to 0} v_{app} = v$$

*while $v$ is the exact game value of the matrix game with $P \in \mathbb{R}^{1\times m}$ strategy set and, $||P|| = \max\limits_{1\le j\le m!} \max\limits_{1\le i\le m} |P_{est}^{j}(i) - P(i)|$.*

**Proof.** Let $p_{\max}$ and $p_{\min}$ be determined with respect to the Theorem 2.7 and Theorem 2.8 for $k = 1$ (or Theorem 3.1 and Theorem 3.2 in [19]). Then, we can determine the rest of the strategy set by obeying Remark 2.11 and, assume that $P_{est} = \{p_{\min}, p_{\max}, p_1, p_2, \dots, p_{m-2}\} \in \mathbb{R}^{1\times m}$ represents one of the possible strategy set. Therefore we have $m!$ different orders of the elements of the strategy set $P_{est}$. This implies that we have $m!$ alternatives for the strategy set which can be considered through the approaches (*i.e.* $P_{est}^{1} = \{p_{\min}, p_{\max}, p_1, p_2, \dots, p_{m-2}\}, P_{est}^{2} = \{p_{\max}, p_{\min}, p_1, p_2, \dots, p_{m-2}\}, \dots, P_{est}^{m!} = \{p_{m-2}, \dots, p_2, p_1, p_{\max}, p_{\min}\}$). In short, we show that all the possible alternatives of the strategy set by $P_{est}^{j} = \{p_{\min}^{j}, p_{\max}^{j}, p_1^{j}, p_2^{j}, \dots, p_{m-2}^{j}\}$ while $j = 1, 2, 3, \dots, m!$. Now, let's show the estimation errors with $e^{j} = v_{app}^{j} - v$, then

$$\begin{aligned} e^{j} &= v_{app}^{j} - v \\ &= \left[p_1^{j} a_{1j} + p_2^{j} a_{2j} + \dots + p_{(m-1)}^{j} a_{(m-2)j} + p_{\min}^{j} a_{(m-1)j} + p_{\max}^{j} a_{mj}\right] \\ &\quad - \left[p_1 a_{1j} + p_2 a_{2j} + \dots + p_{(m-2)} a_{(m-2)j} + p_{m-1} a_{(m-1)j} + p_m a_{mj}\right] \\ &= \left[\left(p_1^{j} - p_1\right) a_{1j} + \left(p_2^{j} - p_2\right) a_{2j} + \dots + \left(p_{(m-1)}^{j} - p_{m-2}\right) a_{(m-2)j}\right. \\ &\quad \left. + \left(p_{\min}^{j} - p_{m-1}\right) a_{(m-1)j} + \left(p_{\max}^{j} - p_m\right) a_{mj}\right] \end{aligned}$$

can be obtained for any one of the possible strategy set $P_{est}^{j}$ obtained by MN approaches where $P = \{p_1, p_2, \dots, p_m\}$ is the exact strategy set of matrix game A. Thus,

$$\begin{aligned} \lim_{||P||\to 0} |v_{app}^{j} - v| &= \lim_{||P||\to 0} \Big| (p_1^{j} - p_1) a_{1j} + (p_2^{j} - p_2) a_{2j} + \ldots + (p_{(m-1)}^{j} - p_{m-2}) a_{(m-2)j} \\ &\quad + (p_{\min}^{j} - p_{m-1}) a_{(m-1)j} + (p_{max}^{j} - p_m) a_{mj} \Big| \\ &\le a_{1j} \cdot 0 + a_{2j} \cdot 0 + \ldots + a_{(m-2)j} \cdot 0 + a_{(m-1)j} \cdot 0 + a_{mj} \cdot 0 \\ &= 0 \end{aligned}$$

by Sandwich theorem since $0 \le |p_i^{j} - p_i| \le ||P||$ for all $i = 1, 2, \dots m$ and $j = 1, 2, \dots, m!$ where $||P|| = \max\limits_{1\le j\le m!} \max\limits_{1\le i\le m} |P_{est}^{j}(i) - P(i)|$. □

**Remark 3.2.** It is clear from Theorem 3.1 that, by MN approaches, we can find possible strategy set $P_{est}$ that is sufficiently close to actual strategy set $P$ of zero-sum matrix game $A$ if $||P|| \to 0$ is possible. On the other hand, there are some zero-sum matrix games whose maximum and minimum strategies are not close enough to each other for each $\epsilon > 0$ such as $p_{\max} - p_{\min} > \epsilon$ (for instance, see Example 4 in Section 4). This implies that the norm of $P$ may never approach to 0 (i.e. $\|P\| \ge (p_{\max} - p_{\min}) \nrightarrow 0$). In these cases, the following corollary guarantees the existence of at least one strategy set obtained by MN approaches where the corresponding $v_{app}$ converges to $v$.

**Corollary 3.2.1.** *Let $A$ be real valued $m \times n$ payoff matrix, where $a_{ij} > 0$ for all $1 \le i \le m$ and $1 \le j \le n$, of a two person zero sum matrix game. Then, there exists at least one strategy set, $P_{est}^{j_0}$ where $1 \le j_0 \le m!$, obtained by MN approaches such that the corresponding $v_{app}$ approaches to actual game value $v$ of zero-sum matrix game $A$ while $||P|| \to 0$ is not possible, in general.*

**Proof.** Assume that $A$ be a zero-sum matrix game whose maximum and minimum strategies are not close enough to each other for each $\epsilon > 0$ such as $p_{\max} - p_{\min} > \epsilon$, thus $\|P\| \nrightarrow 0$. On the other hand, the Theorem 2.7 and Theorem 2.8 guarantee that there are some $p_{\max}^{j}$ and $p_{\min}^{j}$ such that $p_{\max} - p_{\max}^{j_0} < \epsilon$ and $p_{\min} - p_{\min}^{j_0} < \epsilon$ for all $\epsilon > 0$ and some $j_0 \in [1, m!]$. This implies that there exists at least one strategy set $P_{est}^{j_0}$ obtained with MN approaches where the norm of the corresponding $P$ converges to 0 such as $||P^{j_0}|| = \max\limits_{1\le i\le m} |P_{est}^{j_0}(i) - P(i)| \to 0$. Consequently,

$$\lim_{||P^{j_0}||\to 0} v_{app} = v$$

by Theorem 3.1. □

## 4. Illustrations

In this section, we aim to show the consistency of the extended Matrix Norm approach for the bimatrix game. For instance, we deal with one of the famous examples in game theory; the battle of sexes and, another explanatory bimatrix game. The reader may refer to [19,21] to find more examples about game creation and solution of a two-person matrix game with details. In the following two examples, we first separate the bimatrix game into two matrix games and apply Matrix Norm and Extended Matrix Norm approaches directly to show the effects of the refinement obtained by the extended MN approach explicitly under the statements of the related theorems. The last two examples are only considered to demonstrate the consistency of the results obtained for the convergence of the Matrix Norm method in Theorem 3.1 and Corollary 3.2.1.

**Example 1.** Consider the two-person nonzero-sum game with the bimatrix presented in [18],

$$A = \begin{bmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{bmatrix}$$

and the related payoff matrices of Player I and Player II, respectively, are

$$A_1 = \begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

The given bimatrix game values in [18] are $(v^1, v^2) = (2.5, 2.6)$ and the mixed strategy sets are $S_1 = (0.5, 0.5)$ and $S_2 = (0.6, 0.4)$ for Player I (row player) and Player II (column player), respectively.

We now evaluate the interval for the game value, $v^k$, with extended MN methodology presented in this study. To show the advantage part of this paper contributions, we first obtain intervals for each player by using Theorem 2.5 as follows:

$$\frac{3}{7} \le v^1 \le 5 \tag{1}$$

$$\frac{4}{7} \le v^2 \le 6 \tag{2}$$

Then, we use Theorem 2.9 to obtain the boundaries for the game values of each player as

$$1 \le v^1 \le 5$$

$$\frac{5}{6} \le v^2 \le 5$$

At this stage, in order to use Theorem 2.7 directly, we need to choose a dummy game value for the players from the corresponding interval as it is done in the solution that of Example 3 in [19], as an example choose them as an average of each interval as $v^1 = 3$ and $v^2 = 2.92$. Now, if we apply Theorem 2.7 for each player under the assumptions, then we have $p^1_{\max} \ge L_1 = max\{\frac{1-\frac{3}{5}}{2-1}, \frac{3}{5}\} = 0.6$ and $p^2_{\max} \ge L_2 = max\{\frac{1-\frac{2.92}{6}}{2-1}, \frac{2.92}{5}\} = 0.58$. Then, we select $p^1_{\max} = 0.65$ and $p^2_{\max} = 0.6$ in the light of the inequalities for $p^k_{\max}$ where $k = 1, 2$. It is clear that $p^1_{\min} = 0.35$ and $p^2_{\min} = 0.4$ since $p^k_{\max} + p^k_{\min} = 1$. Therefore, the mixed strategy sets in this scenario can be preferred as $S_1 = (0.35, 0.65)$ and $S_2 = (0.6, 04)$.

Moreover, we can obtain a better interval for the game values after the above evaluation by using Theorem 2.10 as, $0.35 \times 5 \le v^1 \le 0.65 \times 5$ and $0.4 \times 6 \le v^2 \le 0.6 \times 5$.

$$1.75 \le v^1 \le 3.25 \tag{3}$$

$$2.4 \le v^2 \le 3 \tag{4}$$

Furthermore, if we compare the intervals of the game value for each player obtained by Theorem 2.5 and Theorem 2.10, Eqs. (1) and (2) to Eqs. (3) and (4), respectively, we can see that Theorem 2.10 refines the boundaries of the game value. This iterated application of the MN method (Theorem 2.9 and Theorem 2.10, respectively) shows that we obtain a better interval for the game values. This refinement is one of the advantages parts of the extended MN approach for each player compared with MN approach in [19].

By considering the refinement interval for each game value, if we apply Theorem 2.7, we may obtain better strategy elements as $p^1_{\max} \ge L_1 = max\{\frac{1-\frac{2.5}{5}}{2-1}, \frac{2.5}{5}\} = 0.5$ and $p^2_{\max} \ge L_2 = max\{\frac{1-\frac{2.7}{6}}{2-1}, \frac{2.7}{5}\} = 0.55$. The refined strategy sets can be reconsidered as $S_1 = (0.45, 0.55)$ and $S_2 = (0.6, 0.4)$.

Finally, in order to see the estimation errors, we calculate $v^k_{app}$ for the players, $v^1_{app} = (3 \times 0.45) + (2 \times 0.55) = 2.45$ and $v^2_{app} = (3 \times 0.6) + (1 \times 0.4) = 2.6$. Hence, the errors in the approximated game values are $e_1 = |2.5 - 2.45| = 0.05$ and $e_2 = |2.6 - 2.6| = 0$. One may create other scenarios with a different selection of a dummy game value and strategies which might result in smaller errors in the approximated game values.

**Example 2.** A famous example of a bimatrix game is the battle of sexes. We have a new married couple and they want to go somewhere together at night. The man likes football (option F), but the woman wants to go theater (option T). If the couple sees the theater, the payoff for the man is 1 and it is 4 for the woman. If the couple goes to the stadium, then the payoffs are vice versa, that are, 4 for the man and 1 for the woman. If the couple joins different activities, they both are unhappy. The bimatrix of the game is

$$A = \begin{bmatrix} & T & F \\ T & (1,4) & (0,0) \\ F & (0,0) & (4,1) \end{bmatrix}$$

First, we consider the game for the man's side. Therefore, the respective payoff matrices for the man and woman are as follows

$$A_1 = \begin{bmatrix} & T & F \\ T & 1 & 0 \\ F & 0 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} & T & F \\ T & 4 & 0 \\ F & 0 & 1 \end{bmatrix}$$

This game may be considered as cooperative game. The strategy set $S_1 = \{0.8, 0.2\}$ and the game value $v_1 = 0.8$ are given in [17]. Now, we use MN approach to get estimations. By using the Theorem 2.5, we have $\frac{1}{0+4} \leq v^1 \leq \frac{0+4}{1+0}$. Simply,

$$0.25 \leq v^1 \leq 4 \tag{5}$$

Now, if we apply extended MN method directly using Theorem 2.9, we will have the following:

$$0.25 \leq v^1 \leq 1$$

In order to find the lower/upper boundaries for the minimum-maximum elements of the strategies, we use Theorem 2.7 for $m = 2$, $p^1_{\max} \geq L_1$ where $L_1 = \max\{1 - \frac{0.8}{0+4}, \frac{0.8}{1}\} = 0.8$. Similarly, we apply the same theorem for $p^1_{\min} \leq U_1$ where $U_1 = \min\{1 - \frac{0.8}{1+0}, \frac{0.8}{4}\} = 0.2$. Hence, we get $p^1_{\max} \geq 0.8$ and $p^1_{\min} \leq 0.2$.

Moreover, if we use Theorem 2.10 for $p^1_{\max} = 0.81$ and $p^1_{\min} = 0.19$, we have $(0.19)4 \leq v_{app} \leq (0.81)1$. Therefore, according to Remark 2.11, the optimized boundaries for the game value is:

$$0.76 \leq v^1 \leq 0.81 \tag{6}$$

Thus, the results in Eqs. (5) and (6) contain the actual game value. Furthermore, the refinement for the boundary of the game value can be seen explicitly from the comparison of Eqs. (5) and (6). One can obtain a similar interval for the woman's situation by using the payoff matrix $A_2$, as well.

**Example 3.** Let $A$ be a payoff matrix of Rock-Paper-Scissors game (RPS) with the strategy set $P = \{0.33, 0.33, 0.33\}$ and the game value is $v = 0$ [20]:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

It is obvious that $p_{\max} = p_{\min} = 0.33$ that are maximum and minimum strategies of RPS game are sufficiently close to each other. Therefore, this is one of the good example to show the convergence of MN approaches, while $||P|| \to 0$ is possible, under the light of Theorem 3.1. directly. First, we need to perturbate the payoff matrix A of RPS game by Proposition 2.7 in [19] in order to get rid of the negative entries and reduce it into the applicable form of the Matrix Norm approaches. For this purpose, we select any $x = 3$ that makes all entries positive and ease our calculations since the strategy set is preserved and the game value is only perturbed 3 units [19]. Then, we obtain the following perturbed payoff matrix with the same strategy set $P$ above while the perturbed game value is $v_p = 0 + 3 = 3$,

$$A_p = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Then, we calculate the corresponding matrix norms to use the theorems as $||A_p||_\infty = 9$, $||B_p||_\infty = 9$, $||A_p||_1 = 9$ and $||B_p||_1 = 9$. Next, we use Theorem 2.5 and obtain the boundaries for the game value, $1 \leq v_p \leq 9$ and choose $v_{dummy} = 5$ as an example. Later, we evaluate upper and lower bounds for $p_{\min}$ and $p_{\max}$ by Theorem 2.7, respectively, find the following boundaries: $p_{\min} \leq U, U = \min\{0.22, 0.56\}$, and select $p_{\min} = 0.21$ and $p_{\max} \geq L, L = \max\{0.22, 0.56\}$, and choose $p_{\max} = 0.57$. Finally, we obtain all possible strategy sets by obeying Remark 2.11 as follows:

$P^1 = \{0.22, 0.21, 0.57\}$ and $v_{app1} = 2.64$
$P^2 = \{0.22, 0.57, 0.21\}$ and $v_{app2} = 2.65$
$P^3 = \{0.21, 0.57, 0.22\}$ and $v_{app3} = 2.64$
$P^4 = \{0.21, 0.22, 0.57\}$ and $v_{app4} = 2.65$
$P^5 = \{0.57, 0.22, 0.21\}$ and $v_{app5} = 2.64$
$P^6 = \{0.57, 0.21, 0.22\}$ and $v_{app6} = 2.65$

Under the above scenarios, it is clear that $||P|| = 0.24$ is not sufficiently close to zero for all possible strategy sets, therefore most of the approximated game values are not close enough to $v_p = 3$, and the mean error is 0.355. To obtain better convergence results by MN approaches and present the importance of the Theorem 2.10 which is one of the most important contributions of this paper, we refine the boundaries of the game value by Theorem 2.10 and obtain the new interval for the game value as $1.89 \leq v_p \leq 5.13$. Then, we again determine a dummy game value in this refinement interval as

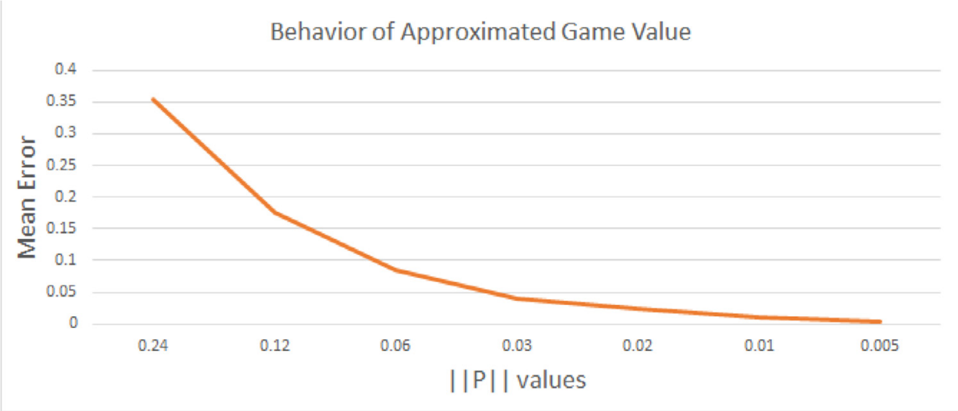


**Fig. 2.** Convergence of MN method: Mean Error vs $||P||$ values.

an example $v_{dummy} = 3.51$. After that, we update $p_{\min}$ and $p_{\max}$ by Theorem 2.7 as $p_{\min} \leq U, U = \min\{0.305, 0.39\}$, select $p_{\min} = 0.3$ and $p_{\max} \geq L, L = \max\{0.305, 0.39\}$, and choose $p_{\max} = 0.39$. Finally, we recreate the possible strategy sets by obeying Remark 2.11 as follows:

$P^1 = \{0.39, 0.30, 0.31\}$ and $v_{app1} = 2.92$
$P^2 = \{0.39, 0.31, 0.30\}$ and $v_{app2} = 2.91$
$P^3 = \{0.31, 0.39, 0.30\}$ and $v_{app3} = 2.91$
$P^4 = \{0.31, 0.30, 0.39\}$ and $v_{app4} = 2.92$
$P^5 = \{0.30, 0.31, 0.39\}$ and $v_{app5} = 2.92$
$P^6 = \{0.30, 0.39, 0.31\}$ and $v_{app6} = 2.91$

In the latter scenario, if we compare the corresponding possible alternative results with the former one, we can state that while $||P||$ gets closer to 0 (i.e. it reduces from $||P|| = 0.24$ to $||P|| = 0.06$), the number of the relatively convergent strategies is almost the same while the respective approximated game value approaches better to the actual game value (i.e. $||P|| = 0.06$ and $v_{app2} - v_p = 0.09$ while the mean error reduces to 0.085). To make it more clear, we consider the different scenarios and analyze the behavior of the approximated solutions in terms of the mean error in Fig. 2.

This figure confirms the results which are promised in Theorem 3.1, and shows the consistency of the MN approach. Clearly, we notice that the following statement

$$\lim_{||P|| \to 0} |v_{app} - v_p| = 0$$

satisfies even though we perturb back the game value $(v_{app} - x) - (v_p - x)$ by the properties of the limit. In the first selection of $v_{pdummy} = 5$, $||P|| = 0.24$ and $v_{app} - v_p = 0.36$ but the elements of the possible strategy sets are not similar to the actual mix strategy set. However, in the second selection of $v_{pdummy} = 3.51$ obtained by Theorem 2.10, we have $||P|| = 0.06$ and $v_{app} - v_p = 0.09$. We see that $||P||$, in this case, is sufficiently close to zero and therefore we obtain a better possible strategy set as stated in Theorem 3.1. If we choose $v_{dummy}$ that results $||P|| = 0$, we can find an approximated game value exactly the same as the actual game value as it is supported by Fig. 2.

**Example 4.** In this example, we want to illustrate what Corallary 3.2.1 states with an application. Let $A$ be real valued $3 \times 3$ payoff matrix of a game with the strategy set $P = \{0.25, 0.5, 0.25\}$ and the game value is $v = 5$ [23]:

$$A = \begin{bmatrix} 5 & 6 & 3 \\ 6 & 3 & 8 \\ 3 & 8 & 1 \end{bmatrix}$$

In order to use the theorems, we need some precalculations such as the related matrix norms: $||A||_\infty = 17$, $||B||_\infty = 14$, $||A||_1 = 17$, and $||B||_1 = 14$. Firstly, we use Theorem 2.5 to find an interval for the game value and choose as a dummy game value from the interval. We calculate the interval as $0.82 \leq v \leq 17$, and select $v_{dummy} = 9$, as an example. Next, we compute upper and lower bound for $p_{\min}$ and $p_{\max}$, respectively, by using Theorem 2.7. Then, we obtain the upper bound for $p_{\min} \leq U, U = min\{0.18, 0.53\}$, and choose $p_{\min} = 0.14$ as an example since $p_{\min} \leq 0.18$,. After, we calculate the lower bound for $p_{\max}$ by using the same theorem as $p_{\max} \geq L, L = max\{0.24, 0.64\}$ and choose it as $p_{\max} = 0.7$. Finally, we decide the element of the strategy set by considering Remark 2.11 and write the possible strategy sets as follows:

$P^1 = \{0.7, 0.14, 0.16\}$ and $v_{app1} = 3.38$
$P^2 = \{0.7, 0.16, 0.14\}$ and $v_{app2} = 3.52$
$P^3 = \{0.16, 0.7, 0.14\}$ and $v_{app3} = 4.18$
$P^4 = \{0.16, 0.14, 0.7\}$ and $v_{app4} = 2.30$

$P^5 = \{0.14, 0.7, 0.16\}$ and $v_{app5} = 4.22$
$P^6 = \{0.14, 0.16, 0.7\}$ and $v_{app6} = 2.40$

Even though, we have some approximated game value that are relatively close to the actual game value with some error, such as 0.82 and 0.78 belonging to the approximated game value obtained by the strategy sets $P_3$ and $P_5$, we know that a better strategy set can be found that is indicated by Theorem 3.1 since $||P|| = 0.45$ does not sufficiently close to zero. Therefore, we need to reconsider the maximum element in the possible strategy sets in order to find a more fitting strategy set such that satisfying $v_{app} - v = 0$.

In the light of these fact, we firstly start with refining the boundaries of the game value by using Theorem 2.10 in order to update $v_{dummy}$ as following, $0.14 \times 17 < v \leq 0.7 \times 12$ and the resulting interval is,

$$2.38 \leq v \leq 8.4 \tag{7}$$

Then, let us select another dummy game value from inequality in (7) as an example $v_{dummy} = 7$ in order to reconsider the maximum element. We use Theorem 2.7 for the computation of upper and lower bound for $p_{\min}$ and $p_{\max}$ again, respectively. We evaluate them as $p_{\min} \leq U, U = \min\{0.25, 0.41\}$ and $p_{\max} \geq L, L = \max\{0.29, 0.5\}$. The upper and lower boundaries are $p_{\min} \leq 0.25$ and $p_{\max} \geq 0.5$. Later, we choose $p_{\max} = 0.52$ and $p_{\min}0.22$. Next, we create $3! = 6$ different strategy sets by the facts stated in Remark 2.11 and write them as follows:

$P^1 = \{0.22, 0.26, 0.52\}$ and $v_{app1} = 3.26$
$P^2 = \{0.22, 0.52, 0.26\}$ and $v_{app2} = 4.96$
$P^3 = \{0.26, 0.22, 0.52\}$ and $v_{app3} = 3.06$
$P^4 = \{0.26, 0.52, 0.26\}$ and $v_{app4} = 5.02$
$P^5 = \{0.52, 0.22, 0.26\}$ and $v_{app5} = 3.58$
$P^6 = \{0.52, 0.26, 0.22\}$ and $v_{app6} = 3.86$

Here, we can see that $\|P\| = 0.28$. Since $||P||$ is not sufficiently close to zero for all possible strategy sets, it means that we can say by Corollary 3.2.1 that, there exists at least one possible distribution of strategies such that almost satisfying the statement in Theorem 3.1 since $||P||$ is not exactly zero in this case. Thus, it is clear that $e = v_{app} - v \to 0$ as $||P|| \to 0$ for the strategy set $P_4$ (or $P_2$) and its game value is $v_{app2} = 5.02$. We can say that if we choose and distribute the elements of the strategy set properly by keeping Remark 2.11 and other related theorems, we can find a strategy set such that $||P|| \to 0$, and therefore we can find the actual game value by the MN methods while $v_{app} \to v$.

## 5. Conclusions and future study

We adapt and apply the new aspect for the solution of the zero-sum matrix games using matrix norms to the nonzero-sum bimatrix games by defining the new notation for the bimatrix games. We state and prove theorems that include the inequalities only depending on $1 - norm$ and $\infty - norm$ of the payoff matrix of each player, for the game value of the bimatrix games. Moreover, we provide the lower and upper boundaries for the maximum and the minimum elements of the strategy set, respectively. In addition to these, we take the boundaries for the game value in the literature one step further for both zero and nonzero-sum matrix games. Thus, we succeed to obtain an improved interval for the game value. Likewise, we readjust the min-max theorem for the bimatrix game.

As a natural consequence of these theoretical frameworks, the approximated solution of the nonzero-sum bimatrix game for each player may be obtained faster since the MN approach does not involve any linear programming or any other computational methods and, the methodology gives the approximated result without dealing with the solution of any equations system. This is one of the strongest parts of this extension for the bimatrix games. Additionally, we analyze the convergence of the MN method (and also the EMN method) in detail and present it as a theorem. In addition, we state a corollary corresponding to the convergence theorem and show the existence of the solution for the Matrix Norm approach. Finally, we comprehensively illustrate the application of the Extended Matrix Norm method in the first example. Additionally, we explain the usage of the method for the game value by considering one of the famous problems, that is battle of sexes, in game theory. The results of these exemplifications confirm that the extended MN approach is also consistent with the literature. Furthermore, and most importantly, the comparisons of the intervals obtained for game values present the advancements of the approach on the boundaries, explicitly. We believe that these improvements may play an important role in the applications and usage of the nonzero-sum bimatrix games. In addition to these, we consider two more examples and illustrate the convergence of the MN method (and also the EMN method) by using the presented theorem.

Although the MN/EMN method works well for the zero-sum and nonzero-sum matrix games, it may be hard to determine and distribute the strategy set elements in an appropriate order when we consider the large-scale matrix games (i.e. $A \in \mathbb{R}^{10\times10}$ or $A \in \mathbb{R}^{100\times100}$, etc.). We also work on this problem and try to solve this issue via artificial intelligence applications of the EMN method in [24]. On the other hand, we think that of Extended Matrix Norm approach may use for finding Nash equilibrium after these developments and extensions for the solution of the zero-sum matrix game and nonzero-sum bimatrix game. The possibility of an algorithm involving the Extended Matrix Norm approach to finding the Nash equilibrium is left for another study which warrants further research.

## Data Availability

Extended Matrix Norm Method: Applications to Bimatrix Games and Convergence Results (Mendeley Data)

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## Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejps.2020.105216.

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