Synchronization Analysis for Stochastic Delayed Multilayer Network With Additive Couplings

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***Abstract*—This paper is concerned with the synchronization of stochastic delayed multilayer networks with additive couplings. Multilayer networks are a kind of complex networks with different layers, which consist of different kinds of interactions or multiple subnetworks. Additive couplings are designed to capture the different layered connections. Based on additive couplings, several sufficient conditions are obtained to guarantee the synchronization of chaotic stochastic delayed coupled multilayer network. More specifically, on one hand, we obtain some sufficient conditions to guarantee that the stochastic multilayer network can be synchronized almost surely without control input. On the other hand, we propose three synchronization schemes by designing controllers. Scheme I: It is assumed that only a part of the nodes are allowed to be controlled directly. Scheme II: Control all nodes of the complex system by using only one controller. Scheme III: Pinning adaptive controller. Finally, an example and its simulations are given to show the effectiveness of our control schemes.**

***Index Terms*—Multilayer network, synchronization, synchronization control.**

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I. Introduction

In 1988, Chua and Yang [1] proposed the theory of cellular neural networks (CNNs). They found many important applications in signal processing and pattern recognition problems, especially in image treatment (see [2]). Since it has great significance in both theory and application, many results on the stability of CNN have been obtained with or without delays. The classical CNN model was described by differential equations

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + I$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\} > 0$, $A \in \mathbb{R}^{n\times n}$ is a constant real matrix. Most of the previous existing results are considering the stability of CNNs. In the real world, time delays occur when the neural networks are usually implemented by VLSI electronic circuits. But, it has been proved that the time delays can cause chaos (see [3]–[7]). Therefore, an important and natural question is about how to control chaos. Pacora and Carroll [8] and Ott *et al.* [9] initially proposed the concept of synchronization and designed a scheme to control chaos. In view of its great practical and theoretic significance, many schemes have been proposed for the synchronization of chaotic systems. These schemes include the adaptive feedback control, the coupling control, the scalar driving method, the manifold-based method, the impulsive control, the adaptive design control, pinning control, and so on. For examples, one can refer to [4], [5], and [9]–[32].

We mentioned that the complex systems are described by the traditional networks in the above works. However, in the real-world applications, the complex network is more complicated than the traditional one. The conventional complex network oversimplifies many important properties and characters in modeling real-world phenomenon. To provide a more general and more realistic description of complex systems in real world, it is very important to propose and study the multilayer networks. Multilayer networks are networks with different layers. Multilayer networks take different kinds of interactions or multiple subnetworks into considerations. There are few papers considering the synchronization of dynamic multilayer networks [33]–[35]. Recently, He *et al.* [35] investigated the synchronization of multiagent systems with different types of interactions. A specific kind of interaction corresponds to a specific layer of the multilayer networks, which forms multiple network topology. Synchronization with additive coupling was proposed in [35]. In fact, on the synchronization problem, a lot of interesting models with additive coupling have been reported in [13]–[15], [28], and [36]. To study the multiple network topologies, He *et al.* [35] proposed and studied the following system with additive couplings:

$$\dot{x}_i(t) = Ax_i(t) + f(t, x_i(t)) + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} D_k \big(x_j(t-\tau) - x_i(t-\tau)\big). \tag{1}$$

However, the realistic environments are usually affected by unpredictable disturbance. This uncertain disturbance can be seen as random, probability, stochastic process, etc. (see [27], [37], [38]). Thus, it is necessary to take the stochastic effects into complex neural networks.

Thus, in this paper, we propose the following stochastic multilayer networks with additive couplings [28], [35], [39]. This is the first paper considering the stochastic multilayer networks. It is assumed that the connections on different layers make positively contribution to the synchronization. The state of each node is evolved according to the following equation:

$$\begin{aligned} dx_i(t) = \Bigg(&Ax_i(t) + f(t, x_i(t), x_i(t-\tau(t))) \\ &+ \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} D_k \big(x_j(t-\tau(t)) - x_i(t-\tau(t))\big)\Bigg) dt \\ &+ u_i(t)dt + g(t, x_i(t), x_i(t-\tau(t)))\mathrm{d}B(t) \\ i = 1, 2, &\ldots, N \end{aligned} \tag{2}$$

where N is the number of coupled nodes, $x_i = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ denotes the state vector, $A \in \mathbb{R}^{n\times n}$ is a constant real matrix, $f(\cdot) \in \mathbb{R}^n$ is a continuous vector function, $c_k \geq 0$ is the strength of coupling, $A^{(k)} = (a_{ij}^{(k)}) \in \mathbb{R}^{N\times N}$ is the outer coupling matrix (adjacency matrix), $D_k \in \mathbb{R}^{n\times n}$ is the kth layer inner coupling matrix and $\tau(t)$ is the transmission delay, $u_i(t) \in \mathbb{R}^n$ represents the control input to ith node and will be designed in the sequel, $B(t)$ is an m-dimensional Brownian motion which is defined on a probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\{B(s), 0 \leq s \leq t\}$, $g(\cdot) \in \mathbb{R}^{n\times m}$ is the noise intensity matrix.

This paper is organized as follows. Model assumptions and preliminary lemmas are given in the next section. We devote ourselves to state our main theorems and their proofs in Section III, which has two sections. One is to obtain some sufficient conditions to guarantee that system (2) can be synchronized almost surely without control input. The other section is to propose three synchronization schemes by designing controllers. Scheme I: It is assumed that only a part of the nodes are allowed to be controlled directly. Scheme II: Control all nodes of the complex system by using only one controller. Scheme III: Pinning adaptive controller. In Section IV, some numerical simulations are given to show the feasibility and effectiveness of our synchronization schemes.

II. Model Description, Assumptions, and Preliminary Lemmas

Notations: I_n is the identity matrix with order n. The matrix $P_1 - P_2$ is said to be positive semi-definite (positive definite, negative semi-definite, and negative definite, resp.) if the symmetric matrices satisfy P_1 and P_2, $P_1 \geq P_2$ ($P_1 > P_2$, $P_1 \leq P_2$, and $P_1 < P_2$). Moreover, $(\Omega, \mathcal{F}, P)$ is some probability space. Let τ be a constant positive real number. The set of all continuous functions from $[-\tau, 0]$ onto $\mathbb{R}^n$ is denoted by $C([-\tau, 0]; \mathbb{R}^n)$. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ denote the set of all bounded and $\mathcal{F}_0$-measurable stochastic variables $\xi = \{\xi(\theta): -\tau \leq \theta \leq 0\}$.

Throughout this paper, we assume that the initial condition to system (2) is described by

$$x_i(t) = \psi_i(t), \quad \psi_i(t) \in C^b_{\mathcal{F}_0}\big([-\tau, 0]; \mathbb{R}^n\big)$$

where $i = 1, 2, \ldots, N$. Moreover, for the matrix $A^{(k)}$, the associated Laplacian matrix $L^{(k)} = (l_{ij}^{(k)})_{N\times N}$ is defined by

$$l_{ij}^{(k)} = -a_{ij}^{(k)}, \text{ for } i \neq j, \text{ and } l_{ii}^{(k)} = \sum_{j=1, j\neq i}^{N} a_{ij}^{(k)}$$

and satisfies $\sum_{j=1}^{N} l_{ij}^{(k)} = 0$ (the diffusion property).

Definition 1: The multilayer complex network (2) is synchronized almost surely if

$$\lim_{t\to+\infty} \big(x_i(t) - x_j(t)\big) = 0, \text{ a.s.}$$

holds for $i, j = 1, 2, \ldots, N$.

Without loss of generality, we choose node 1 as the target node and study the almost sure synchronization of the multilayer complex system (2).

Remark 1: Trivially, as remarked in [39], the almost sure synchronization is equivalent to that

$$\lim_{t\to+\infty} (x_i(t) - x_1(t)) = 0, \text{ a.s.}$$

holds for each $i = 1, 2, \ldots, N$.

Let $e_i(t) = x_i(t) - x_1(t)$, then one obtains the following error system:

$$\begin{cases} \mathrm{d}e_i(t) = (Ae_i(t) + (f(t, x_i(t), x_i(t-\tau(t))) \\ \qquad -f(t, x_1(t), x_1(t-\tau(t)))) \\ \qquad + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} D_k \\ \qquad (e_j(t-\tau(t)) - e_i(t-\tau(t))) + u_i(t) - u_1(t)\big)dt \\ \qquad + (g(t, x_i(t), x_i(t-\tau(t))) \\ \qquad - g(t, x_1(t), x_1(t-\tau(t))))\mathrm{d}B(t) \\ x_i(t) = \psi_i(t) - \psi_1(t), \ t \in [-\tau(t), 0] \end{cases} \tag{3}$$

where $i = 2, \ldots, N$. Note that,

$$\begin{aligned} &\sum_{j=1}^{N} a_{ij}^{(k)} \big(e_j(t-\tau(t)) - e_i(t-\tau(t))\big) \\ &= \sum_{j=1}^{N} a_{ij}^{(k)} e_j(t-\tau(t)) - \sum_{j=1}^{N} a_{ij}^{(k)} e_i(t-\tau(t)) \\ &= -\sum_{j=1}^{N} l_{ij}^{(k)} e_j(t-\tau(t)) = -\sum_{j=2}^{N} l_{ij}^{(k)} e_j(t-\tau(t)) \end{aligned}$$

where $e_1(t-\tau(t)) = 0$. Therefore, the error system (3) can be written in the following form:

$$\begin{cases} \mathrm{d}e(t) = \Big((I_{N-1}\otimes A)e(t) + F(t, e(t), e(t-\tau(t))) \\ \qquad - \sum_{k=1}^{M} c_k\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t)) + U(t)\Big)dt \\ \qquad + G(t, e(t), e(t-\tau(t)))\mathrm{dB}(t) \\ e(t) = \Psi(t),\ \ t\in[-\tau(t), 0] \end{cases} \tag{4}$$

where $e(t) = (e_2^T(t), e_3^T(t), \ldots, e_N^T(t))^T$, $F_i(t, e_i(t), e_i(t-\tau(t))) = f(t, x_i(t), x_i(t-\tau(t))) - f(t, x_1(t), x_1(t-\tau(t))) = f(t, e_i(t)+x_1(t), e_i(t-\tau(t)) + x_1(t-\tau(t))) - f(t, x_1(t), x_1(t-\tau(t)))$, $F(t, e(t), e(t-\tau(t))) = (F_2(t, e_2(t), e_2(t-\tau(t))), \ldots, F_N(t, e_N(t), e_N(t-\tau(t))))^T$, $G_i(t, e_i(t), e_i(t-\tau(t))) = g(t, x_i(t), x_i(t-\tau(t))) - g(t, x_1(t), x_1(t-\tau(t))) = g(t, e_i(t)+x_1(t), e_i(t-\tau(t)) + x_1(t-\tau(t))) - g(t, x_1(t), x_1(t-\tau(t)))$, $G(t, e(t), e(t-\tau(t))) = (G_2^T(t, e_2(t), e_2(t-\tau(t))), \ldots, G_N^T(t, e_N(t), e_N(t-\tau(t))))^T$, $U(t) = (u_2^T(t) - u_1^T(t), \ldots, u_N^T(t) - u_1^T(t))^T$, $\Psi(t) = (\psi_2^T(t) - \psi_1^T(t), \ldots, \psi_N^T(t) - \psi_1^T(t))^T$, and $L_1^{(k)}$ is a matrix formed by removing the first row and the first column of the Laplacian matrix $L^{(k)}$.

For the sake of proving main results, we make the assumptions as follows.

H1: Suppose that there are two constants $M_1 \geq 0$ and $M_2 \geq 0$ such that

$$\begin{aligned} &\|f(t, \xi_1(t), \xi_1(t-\tau(t))) - f(t, \xi_2(t), \xi_2(t-\tau(t)))\|^2 \\ &\quad \leq M_1\|\xi_1(t) - \xi_2(t)\|^2 + M_2\|\xi_1(t-\tau(t)) - \xi_2(t-\tau(t))\|^2 \end{aligned}$$

for any $\xi_1(t), \xi_2(t) \in \mathbb{R}^n$ and $t > 0$. This assumption was given by [24] and [40].

H2: Assume that there are two matrices R_1 and R_2 which are positive definite and

$$\begin{aligned} &\mathrm{trace}\Big((g(t, \xi_1, \eta_1) - g(t, \xi_2, \eta_2))^T(g(t, \xi_1, \eta_1) - g(t, \xi_2, \eta_2))\Big) \\ &\quad \leq (\xi_1 - \xi_2)^T R_1(\xi_1 - \xi_2) + (\eta_1 - \eta_2)^T R_2(\eta_1 - \eta_2) \end{aligned}$$

for any $\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t) \in \mathbb{R}^n$ and $t > 0$. This assumption was motivated by [32].

H3: There are constants $\tau > 0$ and $\rho > 0$ such that $0 \leq \tau(t) \leq \tau$ and $0 \leq \dot{\tau}(t) \leq \rho < 1$, respectively.

Moreover, we may need several lemmas to prove our main result. Assume that $\mathcal{A} = (a_{ij})_{N\times N}$ is an adjacency matrix of some graph where $a_{ij} > 0$ if and only if the jth node can receive information from the ith node and vice versa. Recall that the Laplacian matrix $L = (l_{ij})_{N\times N}$ is defined by

$$l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij},\ \text{ and } l_{ij} = -a_{ij},\ \text{ for } i \neq j$$

and satisfies $\sum_{j=1}^{N} l_{ij} = 0$. Denote L_1 as the matrix obtained by removing the first column and first row of the Laplacian matrix L.

Lemma 1 [39], [41], [42]: The symmetric matrix L has a single eigenvalue 0 and all the other eigenvalues are positive.

Remark 2: We can suppose that the eigenvalues of matrix $L^{(k)}$ are $0 = \lambda_N^{(k)} \leq \lambda_{N-1}^{(k)} \leq \cdots \leq \lambda_1^{(k)}$, $k = 1, 2, \ldots, M$, respectively.

Lemma 2 (Interlacing Theorem [41], [43]): Assume that a real symmetric matrix $L_{N\times N}$ has eigenvalues $\lambda_N(L) \leq \cdots \leq \lambda_1(L)$. Then for any principal submatrix L_m of L obtained by removing m same rows and columns in L, the eigenvalues of L_m interlace with those of L as

$$\lambda_{m+i}(L) \leq \lambda_i(L_m) \leq \lambda_i(L),\ \text{ for any } 1 \leq i \leq N - m.$$

Therefore, for any Laplacian matrix L, $0 \leq \lambda_{\max}(L_1) \leq \lambda_1(L)$ where L_1 is the matrix obtained by removing the first column and first row of L.

Lemma 3: For any positive constant α, two vectors x and y and a square matrix P with compatible dimensions

$$2x^T P y \leq \alpha^{-1}x^T P P^T x + \alpha y^T y. \tag{5}$$

The proof of this lemma is trivial and is omitted.

Finally, to prove our main result, we recall the following LaSalle-type invariance principle [44] for stochastic differential delay equations. Consider the following n-dimensional stochastic differential delayed equation:

$$dx(t) = f(x(t), x(t-\tau), t)dt + \sigma(x(t), x(t-\tau), t)\mathrm{dB}(t). \tag{6}$$

Let $C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ denote the family of all non-negative functions $V(t, x)$ on $\mathbb{R}^n\times\mathbb{R}_+$, which are twice continuously differentiable in x and once in t. For each $V \in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}_+\times\mathbb{R}^n$ to $\mathbb{R}^n$ by

$$\begin{aligned} \mathcal{L}V(t, x, y) = {} & V_t(t, x) + V_x(t, x)f(x, y, t) \\ & + \frac{1}{2}\mathrm{trace}\Big(\sigma^T(x, y, t)V_{xx}\sigma(x, y, t)\Big) \end{aligned}$$

where $V_t(t, x) = [(\partial V(t, x))/\partial t]$, $V_x(t, x) = ([(\partial V(t, x))/\partial x_1], \ldots, [(\partial V(t, x))/\partial x_n])$, $V_{xx} = ([(\partial^2 V(t, x))/(\partial x_i x_j)])_{n\times n}$.

Lemma 4 [44]: Assume that system (6) has a unique solution $x(t, \xi)$ on $t > 0$ for any given initial data $\{x(\theta) : -\tau \leq \theta\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0];\mathbb{R}^n)$. Moreover, both $f(x, y, t)$ and $\sigma(x, y, t)$ are locally bounded in (x, y) and uniformly bounded in t. If there exist a function $V \in C^{2,1}(\mathbb{R}^n\times R_+; R_+)$, $\beta \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\omega_1, \omega_2 \in C(\mathbb{R}^n;\mathbb{R}_+)$ such that for $(t, x, y) \in \mathbb{R}^n\times R^n\times R_+$

$$\mathcal{L}V(t, x, y) \leq \beta(t) - \omega_1(x) + \omega_2(y) \tag{7}$$

$$\omega_1(x) > \omega_2(x), \forall x \neq 0 \tag{8}$$

$$\lim_{\|x\|\to\infty}\ \inf_{0\leq t<\infty} V(t, x) = \infty. \tag{9}$$

Then

$$\lim_{t\to\infty} x(t, \xi) = 0\ \text{ a.s.}$$

for every $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0];\mathbb{R}^n)$.

III. Main Results and Proofs

A. Synchronization Without Control Input

First of all, we will obtain some sufficient conditions to guarantee that system (2) can be synchronized almost surely without control input. That is, to say, we study the almost sure stability of system (4) without control input. In this case, system (4) reduces to the following system of error:

$$\begin{cases} \mathrm{d}e(t) = \Big((I_{N-1}\otimes A)e(t) + F(t,e(t),e(t-\tau(t))) \\ \qquad - \sum_{k=1}^{M} c_k\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t))\Big)dt \\ \qquad + G(t,e(t),e(t-\tau(t)))\mathrm{dB}(t) \\ e(t) \ = \Psi(t),\ \ t\in[-\tau(t),0]. \end{cases} \tag{10}$$

Now, we are in a position to state our result on the synchronization without control input.

Theorem 1: Under assumptions H1–H3, system (10) can be asymptotically stable almost surely, i.e., system (2) can be synchronized almost surely, if there are positive constants $\alpha > 0$, $\beta > 0$ and positive definite matrices $P, R \in \mathbb{R}^{n\times n}$ such that

$$P \le \mu I_n \tag{11}$$

and

$$\begin{pmatrix} \Theta_{1,11} & P & P \\ P & -\alpha I_n & 0 \\ P & 0 & -\frac{\beta}{\sum_{k=1}^{M} c_k} I_n \end{pmatrix} < 0 \tag{12}$$

where $\Theta_{1,11} = PA + A^TP + \rho R + (M_1+M_2)\alpha I_n + \mu(R_1 + R_2) + \beta\sum_{k=1}^{M}(\lambda_1^{(k)})^2 D_k$.

Proof: Choose the Lyapunov functional as follows:

$$V(t,e(t)) = e^T(t)(I_{N-1}\otimes P)e(t) + \int_{t-\tau(t)}^{t} e^T(s)(I_{N-1}\otimes R)e(s)ds.$$

Therefore,

$$\begin{aligned} &\mathcal{L}V(t,e(t),e(t-\tau(t))) \\ &= 2e^T(t)(I_{N-1}\otimes P)\Bigg[(I_{N-1}\otimes A)e(t) + F(t,e(t),e(t-\tau(t))) \\ &\qquad - \sum_{k=1}^{M} c_k\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t))\Bigg]dt \\ &\quad + \mathrm{trace}\big(G^T(t,e(t),e(t-\tau(t)))(I_{N-1}\otimes P) \\ &\qquad G(t,e(t),e(t-\tau(t)))\big) + e^T(t)(I_{N-1}\otimes R)e(t) \\ &\quad - (1-\dot{\tau})e^T(t-\tau(t))(I_{N-1}\otimes R)e(t-\tau(t)) \\ &= e^T(t)\big(I_{N-1}\otimes\big(PA + A^TP + R\big)\big)e(t) \\ &\quad + 2e^T(t)(I_{N-1}\otimes P)F(t,e(t),e(t-\tau(t))) \\ &\quad - 2e^T(t)(I_{N-1}\otimes P)\sum_{k=1}^{M} c_k\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t)) \\ &\quad - (1-\dot{\tau})e^T(t-\tau(t))(I_{N-1}\otimes R)e(t-\tau(t)) \\ &\quad + \mathrm{trace}\big(G^T(t,e(t),e(t-\tau(t)))(I_{N-1}\otimes P) \\ &\qquad G(t,e(t),e(t-\tau(t)))\big). \end{aligned}$$

Meanwhile, for any $\alpha > 0$, using Lemma 3

$$\begin{aligned} &2e^T(t)(I_{N-1}\otimes P)F(t,e(t),e(t-\tau(t))) \\ &= \sum_{i=2}^{N} 2e_i^T(t)PF_i(t,e_i(t),e_i(t-\tau(t))) \\ &\le \sum_{i=2}^{N}\Big(\alpha^{-1}e_i^T(t)PP^Te_i(t) \\ &\qquad + \alpha F_i^T(t,e_i(t),e_i(t-\tau(t)))F_i(t,e_i(t),e_i(t-\tau(t)))\Big) \\ &\le \alpha^{-1}e^T(t)\big(I_{N-1}\otimes PP^T\big)e(t) \\ &\quad + M_1\alpha e^T(t)e(t) + M_2\alpha e^T(t-\tau(t))e(t-\tau(t)). \end{aligned}$$

Similarly, for any given $\beta > 0$

$$\begin{aligned} &-2e^T(t)(I_{N-1}\otimes P)\sum_{k=1}^{M} c_k\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t)) \\ &= \sum_{k=1}^{M} c_k\cdot 2e^T(t)(I_{N-1}\otimes P)\Big(-L_1^{(k)}\otimes D_k\Big)e(t-\tau(t)) \\ &\le \sum_{k=1}^{M} c_k\Big(\beta^{-1}e^T(t)(I_{N-1}\otimes P)(I_{N-1}\otimes P)e(t) \\ &\qquad + \beta e^T(t-\tau(t))\Big(L_1^{(k)}\otimes D_k\Big)\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t))\Big) \\ &= \sum_{k=1}^{M} c_k\Big(\beta^{-1}e^T(t)\Big(I_{N-1}\otimes P^2\Big)e(t) \\ &\qquad + \beta e^T(t-\tau(t))\Big(L_1^{(k)}\otimes D_k\Big)\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t))\Big). \end{aligned}$$

Moreover, using (11)

$$\begin{aligned} &\mathrm{trace}\big(G^T(t,e(t),e(t-\tau(t)))(I_{N-1}\otimes P)G(t,e(t),e(t-\tau(t)))\big) \\ &= \mathrm{trace}\Bigg(\sum_{i=2}^{N} G_i^TPG_i\Bigg) \\ &\le \lambda_{\max}(P)\sum_{i=2}^{N}\mathrm{trace}\big(G_i^TG_i\big) \\ &\le \lambda_{\max}(P)\sum_{i=2}^{N}\big(e_i^T(t)R_1e_i(t) + e_i^T(t-\tau(t))R_2e_i(t-\tau(t))\big) \\ &= \lambda_{\max}(P)\big(e^T(t)(I_{N-1}\otimes R_1)e(t) \\ &\qquad + e^T(t-\tau(t))(I_{N-1}\otimes R_2)e(t-\tau(t))\big) \\ &\le \mu\big(e^T(t)(I_{N-1}\otimes R_1)e(t) \\ &\qquad + e^T(t-\tau(t))(I_{N-1}\otimes R_2)e(t-\tau(t))\big) \end{aligned}$$

where G_i stands for $G_i(t, e_i(t), e_i(t-\tau(t)))$.

Combining all preceding results, we obtain that

$$\begin{aligned} &\mathcal{L}V(t,e(t)) \\ &\le e^T(t)\Bigg(I_{N-1}\otimes\Bigg(PA + A^TP + R + \alpha^{-1}P^2 + M_1\alpha I_n \\ &\qquad + \beta^{-1}\sum_{k=1}^{M} c_kP^2 + \mu R_1\Bigg)\Bigg)e(t) \\ &\quad + e^T(t-\tau(t))\Bigg(I_{N-1}\otimes(M_2\alpha I_n + \mu R_2 - (1-\dot{\tau})R) \\ &\qquad + \beta\sum_{k=1}^{M}\Big(L_1^{(k)}\otimes D_k\Big)\Big(L_1^{(k)}\otimes D_k\Big)\Bigg)e(t-\tau(t)) \\ &= -e^T(t)\Pi_1e(t) + e^T(t-\tau(t))\Pi_2e(t-\tau(t)) \end{aligned}$$

where

$$\Pi_1 = -I_{N-1} \otimes \left(PA + A^T P + R + \alpha^{-1} P^2 + M_1 \alpha I_n + \beta^{-1} \sum_{k=1}^{M} c_k P^2 + \mu R_1 \right)$$

$$\Pi_2 = I_{N-1} \otimes (M_2 \alpha I_n + \mu R_2 - (1-\dot{\tau})R) + \beta \sum_{k=1}^{M} \left(L_1^{(k)} \otimes D_k \right) \left(L_1^{(k)} \otimes D_k \right).$$

Furthermore, by the assumption H3 and Lemma 2

$$\begin{aligned}\Pi_2 - \Pi_1 &= I_{N-1} \otimes \Big(PA + A^T P + \dot{\tau} R + \alpha^{-1} P^2 + (M_1 + M_2)\alpha I_n + \mu(R_1 + R_2) \Big) \\ &\quad + \beta^{-1} \sum_{k=1}^{M} c_k \left(I_{N-1} \otimes P^2 \right) + \beta \sum_{k=1}^{M} \left(L_1^{(k)} \otimes D_k \right) \left(L_1^{(k)} \otimes D_k \right) \\ &\le I_{N-1} \otimes \Big(PA + A^T P + \dot{\tau} R + \alpha^{-1} P^2 + (M_1 + M_2)\alpha I_n + \mu(R_1 + R_2) \Big) \\ &\quad + \beta^{-1} \sum_{k=1}^{M} c_k \left(I_{N-1} \otimes P^2 \right) + \beta \sum_{k=1}^{M} \left(\lambda_1^{(k)} \right)^2 \left(I_{N-1} \otimes D_k^2 \right) \\ &\le I_{N-1} \otimes \left(PA + A^T P + \rho R + \left(\alpha^{-1} + \beta^{-1} \sum_{k=1}^{M} c_k \right) P^2 + (M_1 + M_2)\alpha I_n + \mu(R_1 + R_2) + \beta \sum_{k=1}^{M} \left(\lambda_1^{(k)} \right)^2 D_k^2 \right).\end{aligned}$$

Using the Schur complete lemma [32], one obtains that $\Pi_2 < \Pi_1$ if the inequality (12) holds. Thus, system (3) is asymptotically stable almost surely, i.e., system (2) without the control input is synchronized almost surely. The proof is completed.

B. Synchronization Controller Design

In this section, we consider the synchronization problem by the method of designing controllers. Some appropriate controllers are designed such that the multilayer complex network can be synchronized almost surely.

Scheme I—It Is Assumed That Only a Part of the Nodes Are Allowed to Be Controlled Directly: Without loss of generality, let the indices of these nodes be $i = 2, 3, \ldots, l+1$, $1 \le l \le N-1$, respectively. Motivated by [18], [32], and [39], we set the controller as

$$u_i(t) = -k_i e_i(t) \tag{13}$$

i.e.,

$$U(t) = -[K_1 \otimes I_n] e(t) \tag{14}$$

where $K_1 = \mathrm{diag}\{k_2, k_3, \ldots, k_{l+1}, 0, \ldots, 0\}$, each $k_i > 0$ is the control gain to be designed. Thus, the corresponding system of error is obtained as

$$\begin{cases} \mathrm{d}e(t) = \Big((I_{N-1} \otimes A)e(t) - [K_1 \otimes I_n]e(t) \\ \qquad + F(t, e(t), e(t-\tau(t))) \\ \qquad - \sum_{k=1}^{M} c_k \left(L_1^{(k)} \otimes D_k \right) e(t-\tau(t)) \Big) dt \\ \qquad + G(t, e(t), e(t-\tau(t)))\mathrm{dB}(t) \\ e(t) = \Psi(t), \ t \in [-\tau(t), 0]. \end{cases} \tag{15}$$

Using similar reasoning as that of Theorem 1, the following theorem can be obtained.

Theorem 2: Under assumptions H1–H3, system (15) is asymptotically stable almost surely, i.e., system (2) is synchronized using the controller (13) almost surely, if there exist positive constants $\alpha > 0$, $\beta > 0$ and positive definite matrices $P, R \in \mathbb{R}^{n \times n}$ such that

$$P \le \mu I_n$$

and

$$\begin{pmatrix} \Theta_{2,11} & I_{N-1} \otimes P & I_{N-1} \otimes P \\ I_{N-1} \otimes P & -\alpha I_{n(N-1)} & 0 \\ I_{N-1} \otimes P & 0 & -\frac{\beta}{\sum_{k=1}^{M} c_k} I_{n(N-1)} \end{pmatrix} < 0$$

where $\Theta_{2,11} = I_{N-1} \otimes (PA + A^T P + \rho R + (M_1 + M_2)\alpha I_n + \mu(R_1 + R_2) + \beta \sum_{k=1}^{M} (\lambda_1^{(k)})^2 D_k) - 2K_1 \otimes P = I_{N-1} \otimes \Theta_{1,11} - 2K_1 \otimes P$.

Scheme II—Control All Nodes of the Complex System by Using Only One Controller: In fact, Scheme II is a special case of Scheme I. Thus, we can set the controller as

$$u(t) = k \sum_{i=2}^{N} e_i(t)$$

i.e.,

$$U(t) = -k(E \otimes I_n)e(t) \tag{16}$$

where E is a square matrix with order $N-1$ and all elements equal to 1, $k > 0$ is the control gain to be determined. Therefore, we have the following system of error:

$$\begin{cases} \mathrm{d}e(t) = \Big((I_{N-1} \otimes A)e(t) - k(E \otimes I_n)e(t) \\ \qquad + F(t, e(t), e(t-\tau(t))) \\ \qquad - \sum_{k=1}^{M} c_k \left(L_1^{(k)} \otimes D_k \right) e(t-\tau(t)) \Big) dt \\ \qquad + G(t, e(t), e(t-\tau(t)))\mathrm{dB}(t) \\ e(t) = \Psi(t), \ t \in [-\tau(t), 0]. \end{cases} \tag{17}$$

Using the controller (16), the following corollary can be obtained from Theorem 2.

Corollary 1: Under assumptions H1–H3, system (17) is asymptotically stable almost surely, i.e., system (2) is synchronized almost surely using the controller (16), if there exist positive definite matrices $P, R \in \mathbb{R}^{n \times n}$ and positive constants $\alpha > 0$, $\beta > 0$ such that

$$P \le \mu I_n$$

and

$$\begin{pmatrix} \Theta_{3,11} & I_{N-1}\otimes P & I_{N-1}\otimes P \\ I_{N-1}\otimes P & -\alpha I_{n(N-1)} & 0 \\ I_{N-1}\otimes P & 0 & -\frac{\beta}{\sum_{k=1}^{M} c_k} I_{n(N-1)} \end{pmatrix} < 0$$

where $\Theta_{3,11} = I_{N-1}\otimes(PA+A^TP+\rho R+(M_1+M_2)\alpha I_n+\mu(R_1+R_2)+\beta\sum_{k=1}^{M}(\lambda_1^{(k)})^2 D_k)-2k(E\otimes P) = I_{N-1}\otimes\Theta_{1,11}-2k(E\otimes P)$.

Scheme III—Pinning Adaptive Controller: To obtain better control performance, we will use the pinning adaptive controller. Let the indices of nodes that are allowed to be controlled be $i = 2, 3, \ldots, l+1$, $1 \le l \le N-1$, respectively. Motivated by [18], [32], and [39], we set the controller

$$u_i(t) = -k_i(t)e_i(t) \tag{18}$$

and updated law as

$$\dot{k}_i(t) = \delta\|e_i(t)\|^2 \tag{19}$$

i.e.,

$$U(t) = -(K(t)\otimes I_n)e(t) \tag{20}$$

where $K(t) = \mathrm{diag}\{k_2(t), k_3(t), \ldots, k_{l+1}(t), 0, \ldots, 0\}$, δ is any positive constant. Hence, one obtains the following complex system of error:

$$\begin{cases} \mathrm{d}e(t) = \Big((I_{N-1}\otimes A - K(t)\otimes I_n)e(t) \\ \qquad + F(t, e(t), e(t-\tau(t))) \\ \qquad - \sum_{k=1}^{M} c_k\Big(L_1^{(k)}\otimes D_k\Big)e(t-\tau(t))\Big)dt \\ \qquad + G(t, e(t), e(t-\tau(t)))\mathrm{dB}(t) \\ \dot{K}(t) = \mathrm{diag}\big\{\delta\|e_2(t)\|^2, \delta\|e_3(t)\|^2 \\ \qquad \ldots, \delta\|e_{l+1}(t)\|^2, 0, \ldots, 0\big\} \\ e(t) = \Psi(t),\ t\in[-\tau(t), 0]. \end{cases} \tag{21}$$

For such system, we obtain the following theorem.

Theorem 3: Under assumptions H1–H3, system (21) is asymptotically stable almost surely, i.e., system (2) is synchronized almost surely using the controller (18), if there are positive constants $\alpha > 0$, $\beta > 0$ and positive definite matrices $P, R \in \mathbb{R}^{n\times n}$ such that

$$P \le \mu I_n \tag{22}$$

and

$$\begin{pmatrix} \Theta_{4,11} & I_{N-1}\otimes P & I_{N-1}\otimes P \\ I_{N-1}\otimes P & -\alpha I_{n(N-1)} & 0 \\ I_{N-1}\otimes P & 0 & -\frac{\beta}{\sum_{k=1}^{M} c_k} I_{n(N-1)} \end{pmatrix} < 0 \tag{23}$$

where $\Theta_{4,11} = I_{N-1}\otimes(PA+A^TP+\rho R+(M_1+M_2)\alpha I_n+\mu(R_1+R_2)+\beta\sum_{k=1}^{M}(\lambda_1^{(k)})^2 D_k)-2K_2\otimes I_n = I_{N-1}\otimes\Theta_{1,11}-2K_2\otimes I_n$.

Proof: Choose the Lyapunov functional as follows:

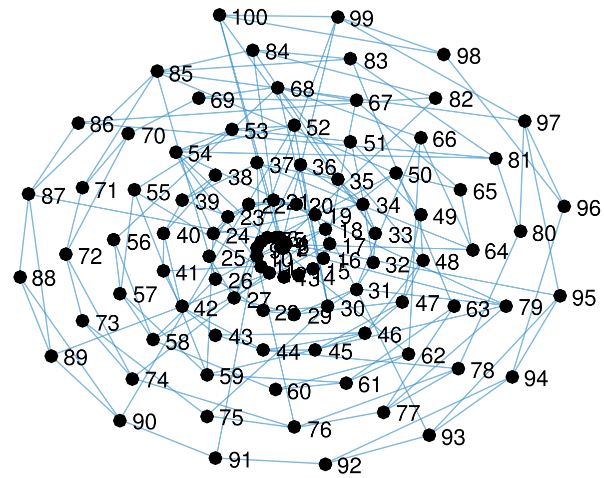
$$V_1(t, e(t)) = V(t, e(t)) + \frac{1}{\delta}\sum_{i=2}^{l}\big(k_i(t)-k^*\big)^2$$

where $V(t, e(t))$ is the same as that in the proof of Theorem 1. Moreover, it is straightforward to find that the differential of $V_1(t, e(t))$ along the trajectories of system (21) satisfies

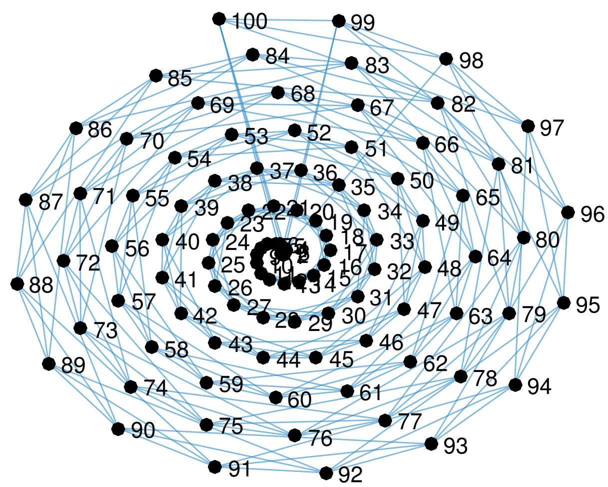
$$\mathcal{L}V_1(t, e(t)) < \mathcal{L}V(t, e(t)) - 2e^T(t)K_2e(t).$$

Therefore, $\mathcal{L}V_1(t, e(t)) < 0$ if (22) and (23) are true. Thus, system (21) is asymptotically stable almost surely, i.e., system (2) with the control input (18) is synchronized almost surely. The proof is completed.

Remark 3: Although Scheme I is easy to be implemented, the control gains are constants and to be designed at the beginning of synchronization control. Therefore, these control gains may be too big and may not be economical. Scheme II use one same controller for all the nodes except node 1. This may be easy to program and be easily applied to small-scale networks. Scheme III use adaptive control gains which are "adaptive" with respect to the evolution of the error. Therefore, they are

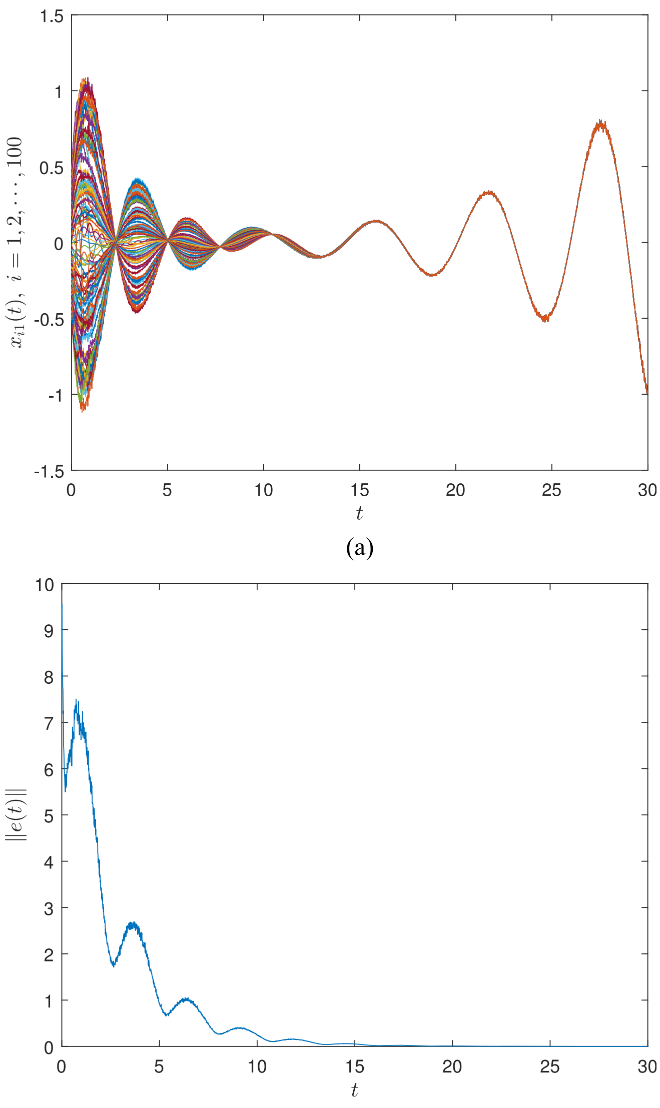


(a)



(b)

Fig. 1. Multilayer network with two layers and 100 nodes. (a) First layer: a Watts–Strogatz small-world graph. (b) Second layer: a regular graph.



(a)

(b)

Fig. 2. Time evolutions of state variables and total errors for the multilayer network. (a) Time evolutions of the state variables for 100 nodes. (b) Time evolution of the total error.

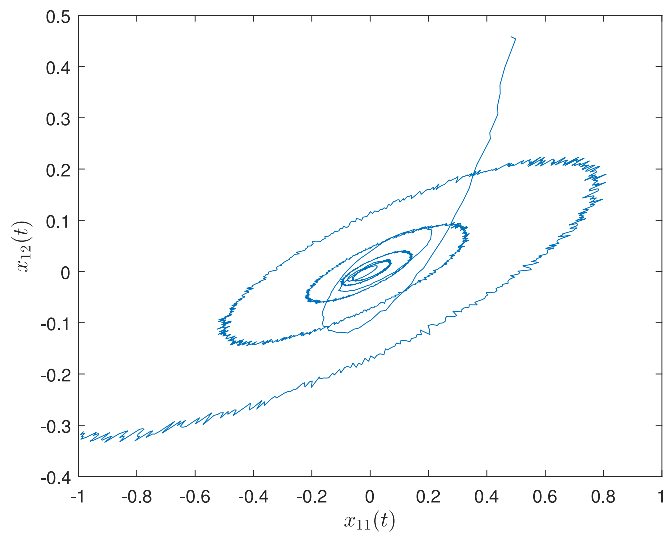


Fig. 3. Evolution of first state variable and second state variable for the node 1.

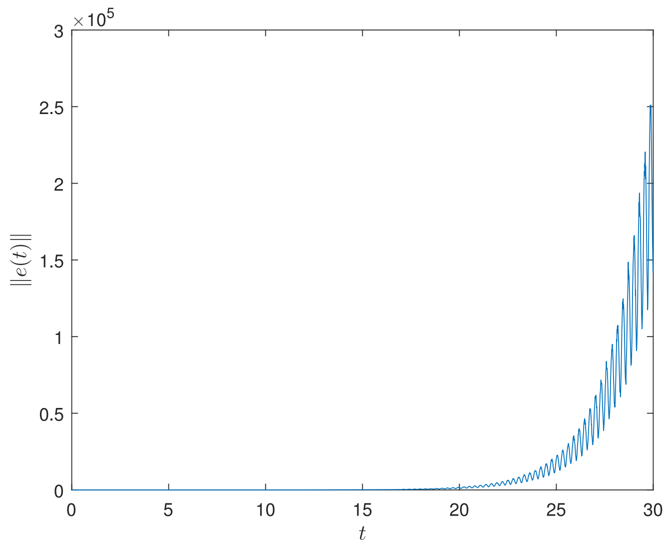


Fig. 4. Evolution of the total error.

economical, but a little bit more difficult to be implemented comparing with Scheme I.

IV. Numerical Examples and Simulation Examples

For each node, we use the same CNN as that provided by [35]

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - m_1 x_1 + f(x_1)) \\ \frac{dx_2}{dt} = x_1 - 2x_2 + x_3 \\ \frac{dx_3}{dt} = -bx_2 \end{cases} \tag{24}$$

where the nonlinear function f is defined by

$$f(x_1) = 0.5(m_1 - m_0)(|x_1 + 1| - |x_1 - 1|)$$

and the parameters $a = 6$, $b = 10/3$, $m_0 = -1/7$, $m_1 = 2/7$. Therefore, in the assumption H1, $M_1 = (27/7)^2$ and $M_2 = 0$.

Example 1: Application to delayed multilayer network with additive couplings and stochastic perturbations.

Consider a multilayer network with $M = 2$ layers and $N = 100$ nodes, as shown in Fig. 1. The first layer is constructed using a Watts–Strogatz small-world graph [45] with initial degree $d = 4$ and the rewiring probability $p = 0.3$, yielding $\lambda_1^{(1)} = 9.5829$. The other layer is a regular graph with degree $d = 6$ and $\lambda_1^{(2)} = 8.6243$. Trivially, the total error function can be defined as

$$\|e(t)\| = \left(\sum_{i=2}^{N}\sum_{j=1}^{n}\big(x_{ij}(t) - x_{1j}\big)^2\right)^{1/2}.$$

Moreover, we set $D_1 = D_2 = I_n$, $\tau(t) = 0.5$ and $g(t, x_i(t), x_i(t-\tau)) = (0.02x_i(t), 0.01x_i(t-\tau))$ in system (2). Thus, $R_1 = 0.02^2 I_n$ and $R_2 = 0.01^2 I_n$ in assumption H2.

With the help of MATLAB LMI Toolbox, solving (11), (12) and choosing $c_1 = c_2 = 0.5$, the delayed multilayer network

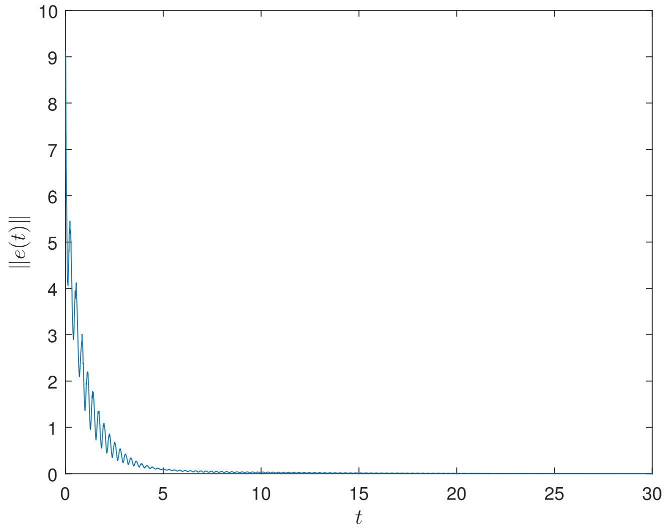


Fig. 5. Evolution of the total error using controller (16) with $k_2 = k_3 = \cdots = k_N = 2$.

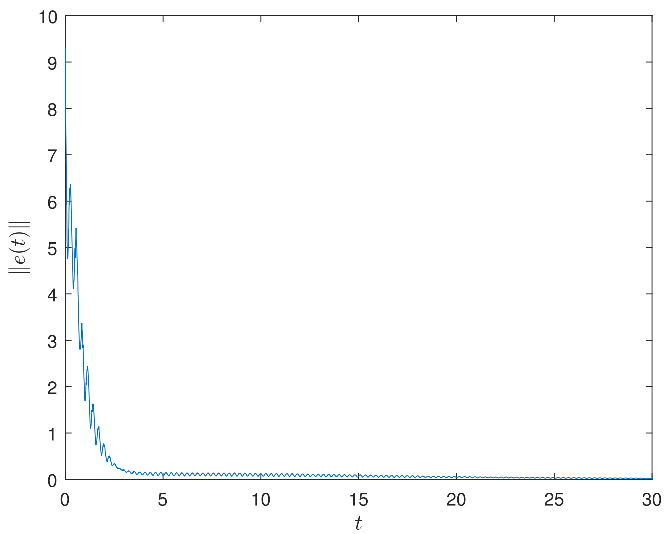


Fig. 6. Evolution of the total error using controller (18) with $\delta = 8$.

achieves synchronization, as shown in Fig. 2. Moreover, Fig. 3 gives the evolution of first state variable and second state variable for the node 1.

However, the network does not achieve synchronization with $c_1 = 0.5$, $c_2 = 0.8$, as shown in Fig. 4. In such case, we design controllers using controllers (14) [or (16)] and (20), thus Theorem 2 (or Corollary 1) and Theorem 3 can be applied, respectively. Figs. 5 and 6 show the time evolutions of the total error under these two controllers, respectively.

V. Conclusion

In this paper, we have investigated the synchronization problem of stochastic delayed multilayer networks with additive couplings, where additive couplings are designed to capture the different layered interactions. In particular, we obtain the sufficient conditions to achieve automatic synchronization (i.e., synchronization without control input). When synchronization does not have control input, we design three control schemes and obtain the sufficient conditions for synchronization. Finally, we give an example to show the effectiveness of our result.

There are still some open interesting problems. For example, we do not obtain the synchronization speed with or without control input. Moreover, the different control inputs are quite restricted in this paper since they are continuous. Discontinuous control inputs, such as the impulsive control input, may also be studied in such model. We leave these questions for future work.

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