Soft Computing Simulations of Chaotic Systems

Erivelton G. Nepomuceno* and Priscila F. S. Guedes†
Control and Modelling Group (GCOM),
Department of Electrical Engineering,
Federal University of São João del-Rei,
São João del-Rei, MG 36307-352, Brazil
*nepomuceno@ufsj.edu.br
†pritl2_guedes@hotmail.com
Alípio M. Barbosa
Department of Electrical Engineering,
Centro Universitário Neon unprecedented,
Belo Horizonte, MG 30431-189, Brazil
alipomonteiro@yahoo.com.br
Matjaž Perc‡‡,§,¶,∥ and Robert Repnik‡‡,∗∗
‡Faculty of Natural Sciences and Mathematics,
University of Maribor, Koruška cesta 160,
SI-2000 Maribor, Slovenia
§CAMTP — Center for Applied Mathematics and
Theoretical Physics, University of Maribor,
Mladinska 3, SI-2000 Maribor, Slovenia
¶Complexity Science Hub Vienna,
Josefstädterstraße 39, A-1080 Vienna, Austria
∥matjaz.perc@um.si
∗∗robert.repnik@um.si
Received January 22, 2019

Soft computing strategies are drawing widespread interest in engineering and science fields, particularly so because of their capacity to reason and learn in a domain of inherent uncertainty, approximation, and unpredictability. However, soft computing research devoted to finite precision effects in chaotic system simulations is still in a nascent stage, and there are ample opportunities for new discoveries. In this paper, we consider the error that is due to finite precision in the simulation of chaotic systems. We present a generalized version of the lower bound error using an arbitrary number of natural interval extensions. The lower bound error has been used to simulate a chaotic system with lower and upper bounds. The width of this interval does not diverge, which is an advantage compared to other techniques. We illustrate our approach on three systems, namely the logistic map, the Singer map and the Chua circuit. Moreover, we validate the method by calculating the largest Lyapunov exponent.

Keywords: Soft computing; chaos; nonlinear dynamics; computer simulation; computer arithmetic; lower bound error; interval arithmetic.

*Author for correspondence
1. Introduction

Soft computing has become a set of tools of great importance in several areas of science and engineering. [Kumar et al. 2013; Zadeh 1994; Bonissone 1997; Dote & Ovaska 2000]. This widespread interest is due to its ability to reason and learn in an environment of uncertainty, approximation and imprecision. These techniques play an important role in nonlinear science, with applications in system identification and modeling. [Kawai & Chen 2004; Kawalek & Schmitt 2014; Sozhamadevi et al. 2013; Liu et al. 2007; Kumar et al. 2014; contra, design, Kalaba & Mihol 2009; Campo et al. 2013; Kurek & Paweleczyk 2014; Nithya et al. 2008; and Sarkar & Mandal 2014.]

Since chaotic systems are so hard to interpret analytically, numerical simulations play a key role in their study. [Parker & Chua 1989]. According to [Shannon 1976], simulation is defined as “the process of designing a computerized model of a system (or process) and conducting experiments with this model for the purpose either of understanding the behavior of the system or of evaluating various strategies for the operation of the system.” Many works have been devoted to investigating the effects of finite precision in the simulation of dynamical systems. [Hammel et al. 1987; Yorke 2004; Luzz 2013; Galil 2014; Nepomuceno 2014; Butusov et al. 2015; Karimov et al. 2015; Nepomuceno et al. 2018a; and Cacciola et al. 2018a]. Chaotic systems implemented with finite precision in digital computers may exhibit totally different dynamical properties when compared to their original version in the continuous setting. [Li et al. 2009]. An important consequence of this has been studied as chaos degradation, which refers to the short cycle length [Cao et al. 2013]. A number of works have focused on chaos degradation [Min et al. 2015; Cao et al. 2015; Hu et al. 2014; Deng et al. 2015; Liu et al. 2011; 2014; Liu & Miao 2017].

Nevertheless, little work can be found in the soft computing literature on finite precision effects in chaotic system simulation. Among these studies, [Cacciola et al. 2018a] have applied soft computing and chaos theory for the prediction of special events in Tokamak reactors. [Yang and Lec 2008] have developed a technique using statistics to soft computing to calculate the most likely forecasted value of a chaotic time series. In addition, Sarkar and Mandal [2014] have used soft computing in key generation for secure communication. They used chaotic systems as the pseudo-random number generator. The reader can refer to [Khondekar et al. 2013] wherein the authors have analyzed various researches already undertaken from the theoretical perspective in the field of soft computing based trend series analysis, characterization of chaos, and theory of fractals. Although these works have been successful in their purposes, the effects of computer finite precision in the simulation of chaotic systems have not yet been carefully considered. In this work, we propose a soft computing simulation of chaotic systems considering uncertainty due to finite precision error. We have presented a generalized version of the lower bound error [Nepomuceno & Martins 2016; 2017; 2019; 2018a; 2018b] using an arbitrary number of natural interval extensions. The lower bound error has been used to simulate a chaotic system with lower and upper bounds. We have shown that the widths of these bounds do not diverge, which is an advantage compared with other techniques based on arithmetic interval. [Moore et al. 2006]. Our approach has been illustrated with three systems: the logistic map [May 1976], Singer map [Aguirregabiria 2005] and Chua’s circuit [Chua 1992; Chua et al. 1992]. The method has been validated using the computation of the largest positive Lyapunov exponent. [Mendes & Nepomuceno 2018; 2019; Kodba et al. 2002].

The rest of the paper is laid out as follows. In Sec. 2 recursive functions, natural interval extension, orbits and pseudo-orbits, and the lower bound error are briefly reviewed. The proposed method based on the lower bound error for an arbitrary number of interval extensions is presented in Sec. 3. To illustrate this approach, examples using the well-known logistic map, Singer map and Chua’s circuit are given in Sec. 4. Section 5 presents the conclusions.

2. Background

In this section, concepts of recursive functions, natural interval extension, orbits and pseudo-orbits, and lower bound error for two pseudo-orbits are briefly described.

2.1. Recursive functions

Let \( n \in \mathbb{N} \), a metric space \( M \subset \mathbb{R} \), the relation

\[ x_{n+1} = f(x_n), \]  

(1)
where \( f : M \to M \) is a recursive function or a map of a state space into itself and \( x_n \) denotes the state at the discrete time \( n \). Given an initial condition \( x_0 \), successive applications of the function \( f \) compute the sequence \( \{x_k\} \). The initial condition \( x_0 \) is called the orbit of \( x_0 \) Gilmore & Lefranc [2012].

2.2. Natural interval extension

The definition of natural interval extension of a function is presented in Moore et al. [2009], and it is as follows.

**Definition 2.1 (Natural Interval Extension).** Let \( f \) be a function of real variable \( x \). A function \( F \) is a natural interval extension of \( f \) if for degenerate interval arguments, \( F \) agrees with \( f \):

\[
F([x, x]) = f(x).
\]

The natural interval extension is achieved by changing the function \( f(x) \) through basic arithmetic operations. When this change is exclusively made by the multiplicative associative property, the natural extensions present equivalent intervals, as shown in Nepomuceno et al. [2017].

2.3. Orbits and pseudo-orbits

The definition of orbit associated to a map is given as in Hammel et al. [1987].

**Definition 2.2 (Orbit).** The true orbit \( \{x_n\}_{n=0}^{N} \) satisfies \( x_{n+1} = f(x_n) \). We have the sequence of values of the map represented by \( \{x_n\} = \{x_1, x_2, \ldots, x_N\} \).

The calculation of an orbit is often performed by a finite precision computer, resulting in a pseudo-orbit. A pseudo-orbit of a map approximates a mathematical orbit in a specific hardware or software. For this reason, the pseudo-orbit cannot be unique Lambers & Sumner [2014].

**Definition 2.3 (Pseudo-Orbit).** Given a map \( x_{n+1} = f(x_n) \), an \( i \)th pseudo-orbit \( \{\hat{x}_n\} \) is an approximation of an orbit given by

\[
\{\hat{x}_n\} = \{\hat{x}_1, \hat{x}_1, \ldots, \hat{x}_n\},
\]

such that

\[
|x_n - \hat{x}_n| \leq \delta_n,
\]

where \( \delta_n \in \mathbb{R} \) is an error bound and \( \delta_n \geq 0 \).

An interval related to each value of a pseudo-orbit is described as:

\[
I_{i,n} = [\hat{x}_{i,n} - \delta_{i,n}, \hat{x}_{i,n} + \delta_{i,n}].
\]

From Eqs. (4) and (5) it is clear that \( x_n \in I_{i,n} \) for all \( i \in \mathbb{N} \).

2.4. The lower bound error

The lower bound error consists of a tool to analyze the error propagation in numerical simulations. Considering only two pseudo-orbits, the lower bound error is described in Theorem [7], the proof of which can be found in Nepomuceno et al. [2017].

**Theorem 1.** Let two pseudo-orbits \( \{\hat{x}_{i,n}\} \) and \( \{\hat{x}_{j,n}\} \) be derived from two natural interval extensions. Let \( \ell_{0,n} = |\hat{x}_{i,n} - \hat{x}_{j,n}|/2 \) be the lower bound error associated to the set of pseudo-orbits \( \Omega = \{\hat{x}_{0,n}, \hat{x}_{1,n}\} \) of a map, then \( \gamma_{0,n} = \gamma_{n} \geq \ell_{0,n} \).

3. Methodology

The key point of this paper is to incorporate the error bound generated at each iteration step in the simulation. We have used the lower bound error to estimate this bound. To assure that the lower bound error is an efficient technique, we have extended the results presented in Nepomuceno & Martins [2014]; Nepomuceno et al. [2017] for an arbitrary number of natural interval extensions. This result is shown in the following theorem.

**Theorem 2.** Let an arbitrary number \( k \in \mathbb{Z}^+ \) of pseudo-orbits be derived from interval extensions. The lower bound error for an arbitrary number of pseudo-orbits is given by

\[
\zeta_n = \max_{i \neq j} \frac{[\hat{x}_{i,n} - \hat{x}_{j,n}]}{2},
\]

where \( i \neq j, i, j \in \mathbb{Z}^+, i \leq k \) and \( j \leq k \). \( n \) stands for each value of a pseudo-orbit.

**Proof.** The proof is conducted by reductio ad absurdum. Conversely, let us assume that it is possible to have a lower bound error described by

\[
\beta_n = \max_{i \neq j} \frac{[\hat{x}_{i,n} - \hat{x}_{j,n}]}{2}.
\]

Then,

\[
I_{i,n} = [\hat{x}_{i,n} - \beta_{i,n}, \hat{x}_{i,n} + \beta_{i,n}].
\]
and
\[ I_{j,n} = [\hat{x}_{j,n} - \hat{\beta}_{j,n}, \hat{x}_{j,n} + \hat{\beta}_{j,n}] \]
for all \( i \) and \( j \). If it is true, considering the two pseudo-orbits, let us say, \( a \) and \( b \), for which we have maximum distance between them, this implies that \( I_{j,n} \cap I_{n} = \emptyset \) which is a contradiction. And that completes the proof. \( \blacksquare \)

Theorem 4 restricted for two pseudo-orbits has been shown in [Nepomuceno et al. 2017]. Then, \( \zeta_n \) can be used as a lower and upper bound for each iteration of a pseudo-orbit, incorporating the error in computational simulations. Here, we show these lower and upper bounds using shadow areas around the pseudo-orbit of chaotic systems.

4. Numerical Experiments

In this section, our approach has been illustrated with three systems: logistic map, Singer map and Chua’s circuit.

4.1. Logistic map

The logistic map is given by [May 1976]:
\[ x_{n+1} = rx_n(1 - x_n), \]
where \( r \) is the control parameter, which belongs to the interval 1 \( \leq r \leq 4 \) and \( x_n \) to the interval 0 \( \leq x_n \leq 1 \).

Let us consider three equivalent interval extensions for the logistic map:
\[ F(X_n) = rX_n(1 - X_n), \]
\[ G(X_n) = r(X_n(1 - X_n)), \]
\[ H(X_n) = X_n(r(1 - X_n)). \]

Equations (8) to (10) are mathematically equivalent. However, they are written slightly differently, as indicated by the underlined terms. Consider the solution of the logistic map with \( r = 3.9 \) and initial condition \( x_0 = 0.1 \). Figure 1(a) shows the result for the interval extension of the logistic map for \( n \in [60, 100] \). After around \( n = 80 \), the pseudo-orbits diverge totally from each other. Figure 1(b) shows the lower bound error for three pseudo-orbits and the associated Lyapunov exponent; the divergence between pseudo-orbits grows exponentially. The Lyapunov exponent was calculated using the method developed in [Mendes & Nepomuceno 2014], furnishing a value of 0.627 bits/n, which is in good agreement with the literature (0.693 bits/n) [Rosenstein et al. 1993].

It is interesting to note that after around 80 iterations the system does not present reliability of the numerical simulation. This is a clear relationship between the Lyapunov exponent and loss of significant bits. In other words, the expression 0.627n − 52.775 approaches zero as \( n \) is around 80.

As we have focused our attention on floating-point representation in a typical 64-bit environment, we started, our simulation with the maximum precision, that is 52 bits (significant). As the number of iterations increases, the simulation loses its precision at a rate of 0.627 bits per iteration. This rationale can be conducted in a similar way for other examples in this paper. For more on this issue, the reader is invited to see [Nepomuceno & Mendes 2013, Nepomuceno et al. 2018a]. As previously reported, the lower bound error describes the error propagation in numerical simulations. Figure 1(a) shows the bound of simulation for the logistic map given by Eq. (8).

4.2. Singer map

The singer map is a one-dimensional system. This map is described as follows [Aguirregabiria 2009]:
\[ x_{n+1} = \mu (7.86x_n - 23.31x_n^2 + 28.75x_n^3 - 13.3x_n^4), \]
where \( x_n \in (0, 1) \), initial condition \( x_0 \in (0, 1) \) and \( \mu \in [0.9, 1.08] \).

Let us see three natural interval extensions for the Singer map:
\[ I(X_n) = \mu (7.86X_n - 23.31X_n^2 + 28.75X_n^3 - 13.3X_n^4), \]
\[ J(X_n) = 7.86X_n - 23.31X_n^2 + 28.75X_n^3 - 13.3X_n^4, \]
\[ K(X_n) = 7.86X_n - 23.31X_n^2 + 28.75X_nX_n - 13.3X_n^4. \]

Consider the solution of the Singer map with \( \mu = 1.07 \) and initial condition \( x_0 = 0.4 \). Figure 1(c) shows the result of the interval extensions for the Singer map for \( n \in [60, 100] \). As can be seen, after \( n = 65 \) the divergence between the pseudo-orbits is
Fig. 1. (a), (c) and (e): Free run simulation of three interval extensions for the logistic map, Singer map and Chua’s circuit, respectively. (b), (d) and (f): Evolution of lower bound error $\zeta_n$ for the pseudo-orbits of the logistic map, Singer map and Chua’s circuit, respectively. The details of the simulation are as follows. (a) Simulation of Eqs. (8)–(10) with $r = 3.9$ and initial condition $x_0 = 0.1$. (c) Simulation of Eqs. (12)–(14) with $\mu = 1.07$ and initial condition $x_0 = 0.4$. (e) Simulation of Eq. (15) with changes proposed in Eqs. (17)–(19). These figures show the efficiency of the proposed technique as they present Lyapunov exponents in good agreement with the literature as summarized in Table 1.
Fig. 2. Representation of uncertainty in the simulation of chaotic systems: (a) logistic map represented by $F(X_n)$ [Eq. (8)], (b) Singer map represented by $I(X_n)$ [Eq. (12)] and (c) Chua’s circuit represented by Eq. (15). For (a) and (b) the $x$-axis is the number of iterations $n$, while in (c), the $x$-axis is time given in seconds. The gray shadow represents the bounds derived from the lower bound error according to Eq. (6). As expected for chaotic systems, the gray shadow grows exponentially. Nevertheless, an advantage of the proposed technique is that the maximum error does not diverge. This is possible, as the error is related to the lower bound error. The reader can refer to [Nepomuceno & Martins, 2017; Nepomuceno et al., 2018] for more details.
shown): Figure 1(b) shows the lower bound error for three pseudo-orbits and the associated Lyapunov exponent calculated as 0.710 bits/n, while a value in literature is 0.690 bits/n [Feng et al. 2017]. Figure 1(b) depicts the pseudo-orbit, where the lower bound error is indicated by a gray shadow.

4.3. Chua’s circuit

Chua’s circuit equations are described as follows [Chua et al. 1992]:

\[
\begin{align*}
C_1 \frac{dv_1}{dt} &= \frac{v_2 - v_3}{R} - i_R(v_1), \\
C_2 \frac{dv_2}{dt} &= \frac{v_1 - v_2 + i_L}{R}, \\
i_L \frac{di_L}{dt} &= -v_2.
\end{align*}
\]

(15)

The current through the nonlinear element, \(i_R(v_1)\), is given by:

\[
i_R(v_1) = \begin{cases} 
  m_0 v_1 + B_p (m_0 - m_1) & v_1 < -B_p, \\
  m_1 v_1 & |v_1| \leq B_p, \\
  m_0 v_1 + B_p (m_1 - m_0) & v_1 > -B_p,
\end{cases}
\]

(16)

where \(m_0\), \(m_1\) and \(B_p\) are the slopes and the breakpoints of the nonlinear element, respectively. Let three arithmetic interval extensions of Eq. (15) be given by (only the equations related to \(v_1\) are shown):

\[
\begin{align*}
A : \frac{dv_1}{dt} &= \frac{1}{C_1} \left( v_3 - v_1 \right) - i_R(v_1), \\
B : \frac{dv_1}{dt} &= \frac{1}{C_1} \left( \frac{1}{R} \left( v_2 - v_1 \right) \right) - i_R(v_1), \\
C : \frac{dv_1}{dt} &= \frac{1}{C_1} \left( \frac{1}{R} \left( v_3 - v_1 \right) \right) - i_R(v_1).
\end{align*}
\]

(17) \hspace{1cm} (18) \hspace{1cm} (19)

The three models [Eqs. (17–19)] were achieved by rearranging the expression that characterizes the voltage in the capacitor \(C_1\). The simulation was performed using the discretization method of Runge–Kutta of fourth order [Quarternoni et al. 2004] and step-size \(h = 10^{-6}\). The component values and constants used are: \(C_1 = 10 \text{nF}, C_2 = 100 \text{nF}, L = 19.2 \text{mH}, R = 1680 \Omega, m_0 = -0.37 \text{mS}, m_1 = -0.68 \text{mS}, B_p = 1.1 \text{V}\). The initial condition \((v_1, v_2, i_L) = (-0.6, 0, 0)\). Figure 1(c) shows the result of the voltage in capacitor \(C_1\). The pseudo-orbits diverge from each other significantly. Figure 1(f) presents the lower bound error for these pseudo-orbits, wherein the same pattern shown in previous cases is also observed, that is, an exponential growth in the divergence of the pseudo-orbits. The literature presents a Lyapunov exponent around 2.901 bits/ms [Lavrov 2014], while the calculated value is 2.199 bits/ms. The representation of uncertainty due to finite precision is given in Fig. 1(c) for this case. As we are using double precision, all 52 bits of precision are lost in approximately 52/2.199 = 23.6 ms. A very similar result has been found by Šalamon and Dogša 2004 (see Fig. 3 of that work for more details).

Table 1 shows the largest positive Lyapunov exponent for each studied system found from the proposed method and those indicated in the literature. Table 1. Calculation of the Lyapunov exponent comparing the proposed method with the values obtained in the literature presented by Rosenstein et al. [1993] (logistic map), by Feng [Feng et al. 2017] (Singer map) and by Lavrov [2014] (Chua’s circuit). For the discrete maps the Lyapunov exponent is measured in bits per number of iterations \(n\). For the continuous systems, it is given in bits per milliseconds (ms).

<table>
<thead>
<tr>
<th>Systems</th>
<th>Literature</th>
<th>Proposed Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logistic map</td>
<td>0.693 bits/n</td>
<td>0.627 bits/n</td>
</tr>
<tr>
<td>Singer map</td>
<td>0.696 bits/n</td>
<td>0.710 bits/n</td>
</tr>
<tr>
<td>Chua’s circuit</td>
<td>2.091 bits/ms</td>
<td>2.199 bits/ms</td>
</tr>
</tbody>
</table>

5. Conclusion

This paper has investigated the generalization of the lower bound error for an arbitrary number of pseudo-orbits to estimate the bounds of the simulation for chaotic systems. The method has been tested in three systems: two discrete maps (logistic and Singer) and a continuous one (Chua’s circuit).

The code used to calculate the error propagation is very simple and it is a low cost procedure. Although, the interval grows exponentially as seen in Figs. 1(b) and 1(c), the bounds do not exceed the limits of the attractor, as it has been already pointed
E. G. Nepomuceno et al.

Acknowledgments
E. G. Nepomuceno was supported by Brazilian Research Agencies: CNPq/ENERG; CNPq (Grant No. 425509/2018-4), FAPEMIG (Grant No. APQ-00870-17), A. M. Barbosa was supported by FAPEMIG (Grant No. APQ-02983-18). M. Perc was supported by the Slovenian Research Agency (Grants J4-9302, J1-9112 and P1-0403).

References


E. G. Nepomuceno et al.


