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Is There a Relation Between Synchronization Stability and Bifurcation Type?

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Synchronization in complex networks is an evergreen subject with many practical applications across the natural and social sciences. The stability of synchronization is thereby crucial for determining whether the dynamical behavior is stable or not. The master stability function is commonly used to that effect. In this paper, we study whether there is a relation between the stability of synchronization and the proximity to certain bifurcation types. We consider four different nonlinear dynamical systems, and we determine their master stability functions in dependence on key bifurcation parameters. We also calculate the corresponding bifurcation diagrams. By means of systematic comparisons, we show that, although there are some variations in the master stability functions in dependence on bifurcation proximity and type, there is in fact no general relation between synchronization stability and bifurcation type. This has important implication for the restrained generalizability of findings concerning synchronization in complex networks for one type of node dynamics to others.

Keywords: Synchronization; master stability function; bifurcation diagram; chaos.

1. Introduction

For studying the complex networks, the main specifications for describing the network are the microscale level (node dynamics) and the macroscale level (the whole network characteristics) [Arenas *et al.*, 2006]. One of the most popular properties at the macroscale level which is attracting the scientists increasingly is the collective behavior of the complex networks [Chavez *et al.*, 2005; Boccaletti *et al.*, 2006]. One of the hottest topics in recent years is the synchronized collective behavior in control and systems, inspired in many applications such as physics, biology, and engineering [Matheny *et al.*, 2014; Totz *et al.*, 2015; Bolhasani *et al.*, 2017; Chowdhury *et al.*, 2019; Parastesh *et al.*, 2019; Li *et al.*, 2019a]. Synchronization is an emergent phenomenon, resulting from the interaction of two or more systems [Boccaletti *et al.*, 2002; Pikovsky *et al.*, 2003; Arenas *et al.*, 2008]. But there has been a larger interest in the synchronization study of networks composed of a large number of elements. An important subject in this area is to determine the stability of the synchronization state [Xie & Chen, 1996; Gao *et al.*, 2006; Russo & Di Bernardo, 2009; Rakshit *et al.*, 2017; Li *et al.*, 2019b]. In 1994, Pecora and Carroll proposed the "Master Stability Function" (MSF) method which gives the necessary conditions for the linear stability of the synchronization manifold in a network of identical oscillators [Pecora & Carroll, 1998, 1999].

The local dynamics of the network elements is an important factor in the emergent collective behavior. Many of the scientists have concentrated on the synchronization of chaotic systems with an emphasis on the large-scale and complex networks [Banerjee *et al.*, 2017; Chen *et al.*, 2018; Li *et al.*, 2019b]. The first chaotic system was discovered in 1963 by Lorenz who found a three-dimensional autonomous system with chaotic attractor during his studies in atmospheric convection [Sparrow, 2012]. After that, Rössler [Rössler, 1976] proposed another three-dimensional chaotic system, simpler than Lorenz equations. Discoveries of the chaotic systems led to a huge flow of research into the analysis of these systems. Some basic tools in this field are the bifurcation diagram and the Lyapunov spectrum. These diagrams can exhibit different dynamics of the system and dynamical transitions with respect to a system parameter [Qi *et al.*, 2005].

The dependence of the system's dynamics to the parameters' values and also the relation of the collective behaviors to the local dynamics of the units of the network, raises this question: Is there any relation between the stability of the synchronized behavior of the elements and their dynamics? To find the answer to this question, in this paper, we calculate the MSF for various values of a system parameter that can be considered as a bifurcation parameter. Then the obtained conditions from MSF solution are compared with the bifurcation diagram. This process is done for four different systems with different bifurcation diagrams.

The rest of the paper is organized as follows. In Sec. 2, the master stability function method is described. The results of calculating the MSF for four systems are presented in Sec. 3. The conclusions are given in Sec. 4.

2. Master Stability Function

In order to calculate the MSF, first consider that the describing equation of the network can be written as follows:

$$\dot{X}_i = F(X_i) + \sigma \sum_j G_{ij} H(X_j), \tag{1}$$

where X_i is the m-dimensional vector of the dynamical variable of the ith oscillator, $F(X_i)$ is the uncoupled dynamical system, and H is the coupling function which determines which node's variable is used in the coupling. In this paper, we consider $H = [1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0]$. σ is the coupling strength and G represents the topology of the connectivity in the network.

Then the relevant variational equation of Eq. (1) is determined as follows:

$$\dot{\xi}_k = [DF + \sigma\gamma_k DH]\xi_k, \tag{2}$$

where ξ_k is the variations on the ith node, and γ_k is an eigenvalue of G, $k = 0, 1, \ldots, N$. Note that in Sec. 3, the parameter d equals $\gamma_{\max} \times \sigma$. The maximum Lyapunov exponent of Eq. (2) is the MSF. If MSF is negative, the network has a stable synchronized state and if it is positive, the synchronization is unstable.

3. Results

To investigate the relationship between system dynamics and synchronization stability of networks, we compare the bifurcation diagram of the system and the MSF diagram in four different systems. The MSF is calculated for different d values with respect to the bifurcation parameter. Therefore, a 2D diagram is presented for the MSF solution. The results of each system are presented in the following.

3.1. *Rössler*

The Rössler system is defined by three differential equations as follows [Yu, 1997]:

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + ay,\\ \dot{z} &= b + z(x - c),\end{aligned} \tag{3}$$

where $b = 0.6$, $c = 6$ and parameter a is considered as the bifurcation parameter in $a = [0, 0.4]$.

To obtain the MSF diagram, the variational equations for the Rössler system, defined in Eq. (3), can be described as follows:

$$\begin{aligned}\dot{\xi}_1 &= -d\xi_1 - \xi_2 - \xi_3,\\ \dot{\xi}_2 &= \xi_1 + a\xi_2,\\ \dot{\xi}_3 &= z\xi_1 + (x - c)\xi_3.\end{aligned} \tag{4}$$

Figure 1(a) shows the bifurcation diagram of the Rössler system according to the parameter a. The system has a period-doubling route to chaos. For plotting the bifurcation diagram, we have used the initial conditions with forward continuation. Figure 1(b) demonstrates the MSF diagram. The

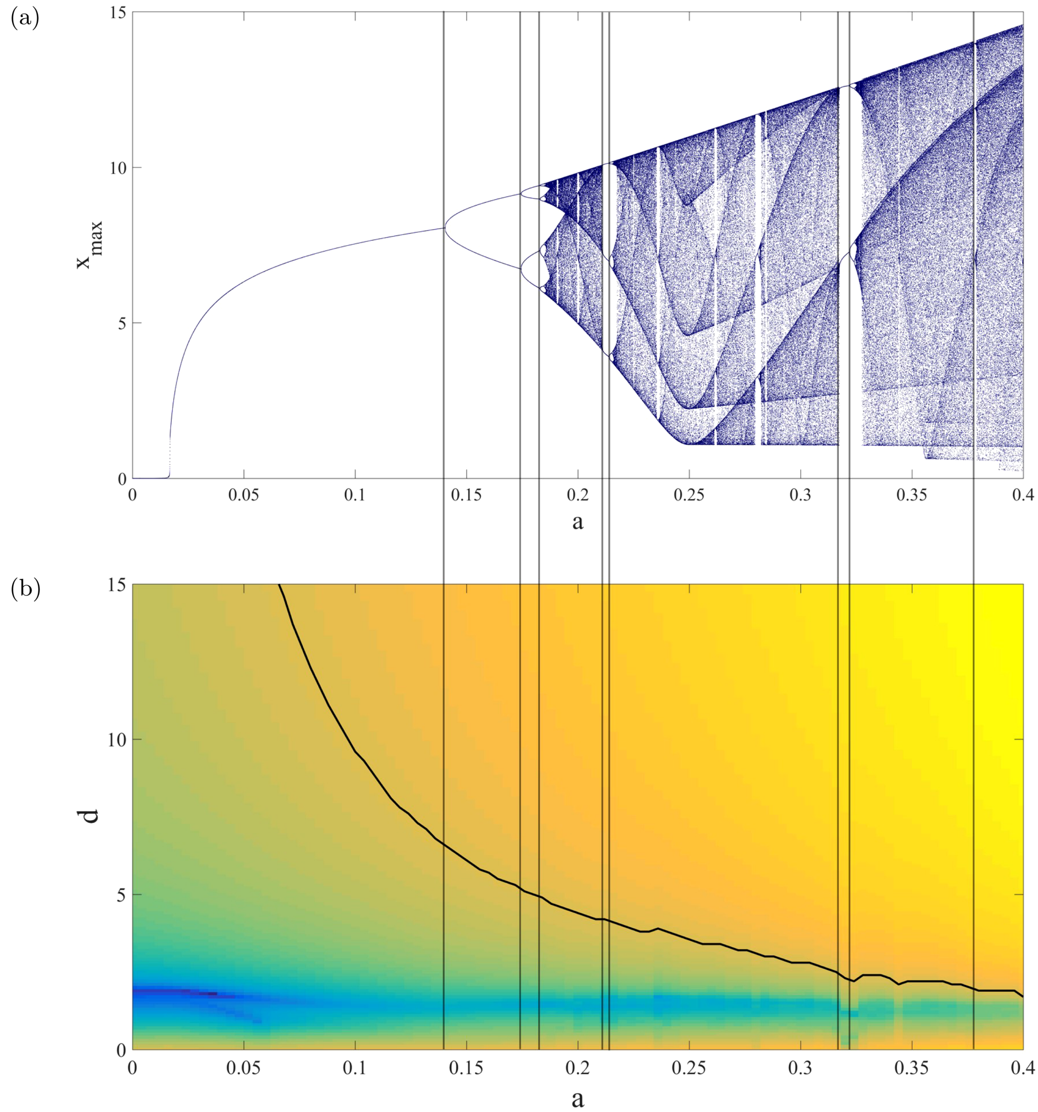

Fig. 1. (a) The bifurcation diagram of Rössler system according to parameter a. (b) The MSF diagram, in the (a, d) plane. The black curve in (b) shows where MSF equals to zero. The vertical lines connect the bifurcation points to their corresponding points in the MSF diagram.

y-axis describes d, and x-axis describes the bifurcation parameter a. The black line shows where the MSF equals to zero. The space above the black line is MSF > 0, where the network is asynchronized and the space below the black line is MSF < 0, where the network is synchronized. Generally, as the bifurcation parameter a grows, the black line has a downward trend. It means that for higher values of a, the network is synchronized for the lower value of d. The trend is smooth and there are no sharp jumps in the line. Therefore, the changes in the dynamics of the elements have no effect on synchronization stability.

3.2. *Hindmarsh–Rose*

The Hindmarsh–Rose neuron model is described by Eq. (5) [Hindmarsh & Rose, 1984]:

$$\begin{aligned}\dot{x} &= y - x^3 + ax^2 + I - z,\\ \dot{y} &= b + cx^2 - y,\\ \dot{z} &= r(s(x - x_1) - z),\end{aligned} \tag{5}$$

where $a = 3$, $b = 1$, $c = -5$, $s = 4$, $x_1 = 1.6$, $I = 3.25$ and $r = [0, 0.5]$ is considered as a bifurcation parameter.

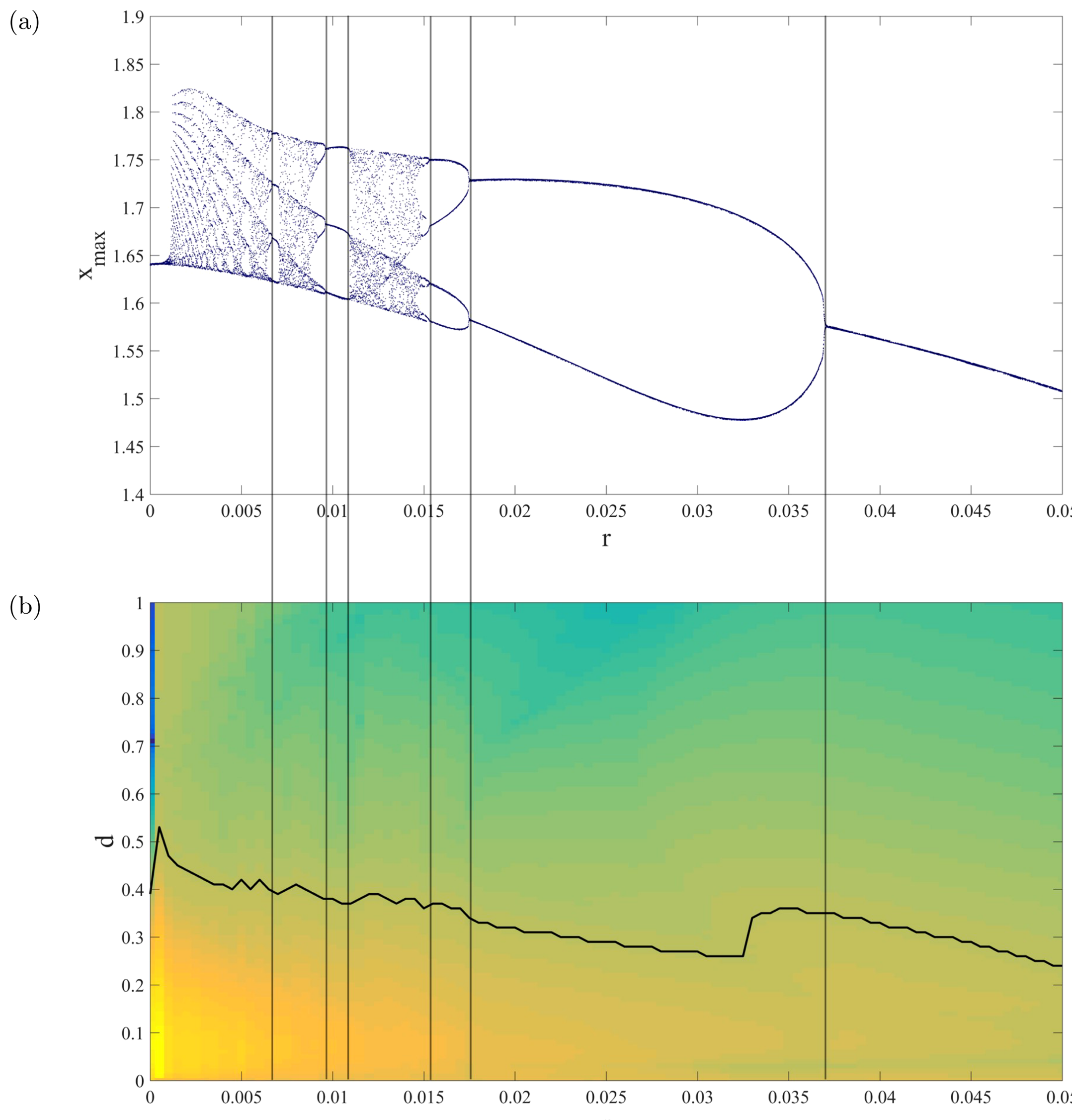

Fig. 2. (a) The bifurcation diagram of Hindmarsh–Rose system according to parameter r. (b) The MSF diagram, in the (r, d) plane. The black curve in (b) shows where MSF equals to zero. The vertical lines connect the bifurcation points to their corresponding points in the MSF diagram.

The variational equations of HR neuron model, based on Eq. (2), are described in Eq. (6):

$$\begin{aligned}\dot{\xi}_1 &= (-d - 3x^2 + 2ax)\xi_1 + \xi_2 - \xi_3,\\ \dot{\xi}_2 &= -2cx\xi_1 - \xi_2,\\ \dot{\xi}_3 &= rs\xi_1 - r\xi_3.\end{aligned} \tag{6}$$

The bifurcation and MSF diagrams are plotted in Fig. 2. Part (a) shows the bifurcation diagram of HR neuronal system with respect to parameter r. The diagram is obtained by the initial conditions with backward continuation. The system demonstrates period-doubling route to chaos. Figure 2(b) describes the MSF diagram of the HR system. The maximum value of the black curve is in $d = 0.53$ and $r = 0.0005$ and minimum value of the black curve is in $d = 0.24$ and $r = 0.05$. There is a jump in $d = 0.26$ and $r = 0.0325$. On the whole, the trend of the black curve is downward, but it has small variations which seems to have no relation with the bifurcations.

3.3. *A 3D chaotic system with specific analytical solution*

As the third system we have chosen a three-dimensional chaotic system with specific analytical solution in e^{-t} that is defined as follows [Faghani *et al.*, 2019]:

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = ax + by - y^2 + xz, \tag{7}$$

where $b = -4$ and a is supposed as the bifurcation parameter in $a = [-7.8, -1.8]$.

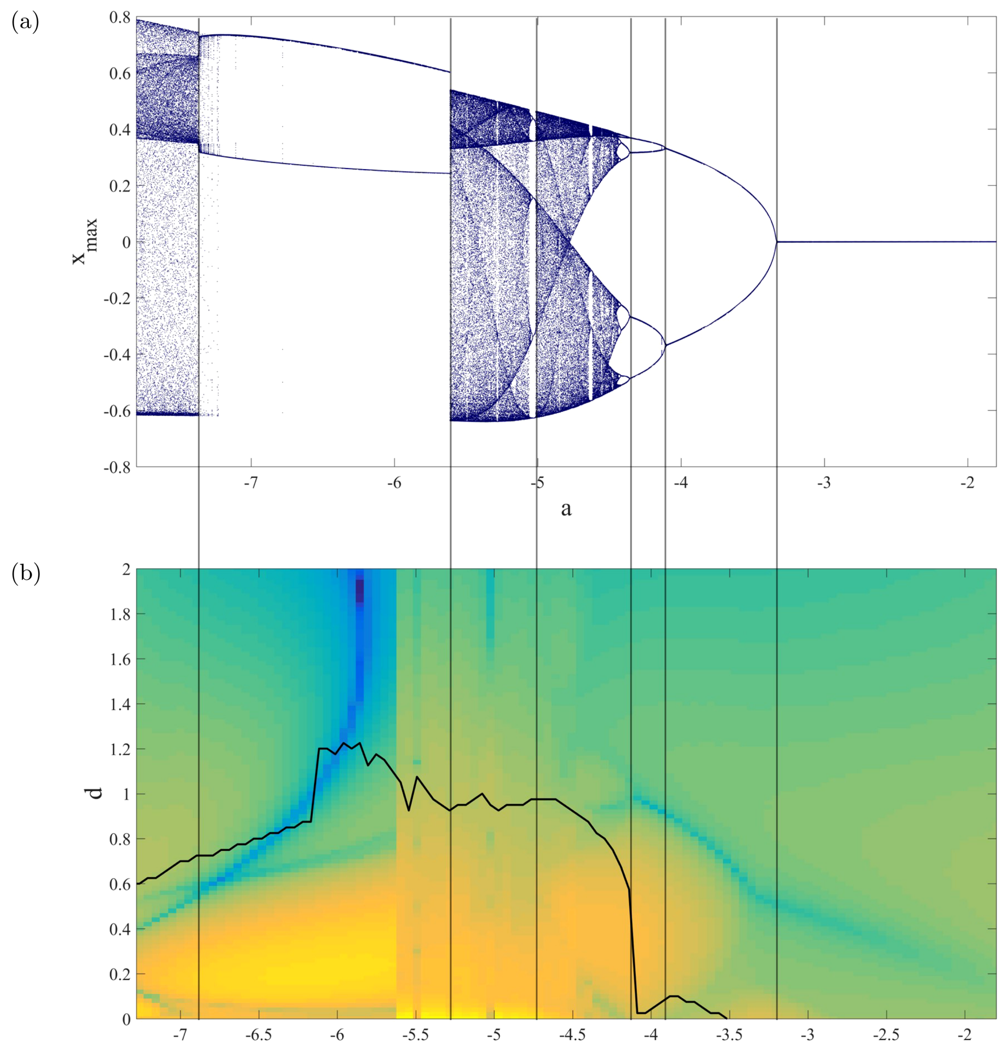

Fig. 3. (a) The bifurcation diagram of third system [Eq. (7)] according to parameter a. (b) The MSF diagram, in the (a, d) plane. The black curve in (b) shows where MSF equals to zero. The vertical lines connect the bifurcation points to their corresponding points in the MSF diagram.

Equation (8) defines the variation of equations of the system:

$$\dot{\xi}_1 = -d\xi_1 + \xi_2, \quad \dot{\xi}_2 = \xi_3, \\ \dot{\xi}_3 = (a+z)\xi_1 + (b-2y)\xi_2 + x\xi_3. \tag{8}$$

The bifurcation and MSF of this system are plotted in Figs. 3(a) and 3(b), respectively. Figure 3(a) depicts that the system has various periodic and chaotic attractors according to the changing bifurcation parameter. The diagram is plotted with the constant initial condition. Figure 3(b) illustrates the MSF diagram with respect to changing bifurcation parameter a and coupling strength d. The general trend of the black curve is downward with growing a, but there are small variations and two big jumps in $[-6.168, 0.88]$ and $[-4.608, 0.975]$. The maximum value of the black curve in $[-5.96, 1.225]$ and the minimum value of the line is in $[-3.464, 0]$. For $a > 3.5$ the MSF does not equal to zero. Comparing Figs. 3(a) and 3(b) implies no special relation between the MSF variation and system bifurcation.

3.4. *A 3D autonomous system without linear terms*

A three-dimensional autonomous system without linear term is specified as follows [Mobayen *et al.*, 2018]:

$$\dot{x} = y^2 + a, \quad \dot{y} = x^2 + y^2 - z^2 + b, \quad \dot{z} = -x^2 + y^2, \tag{9}$$

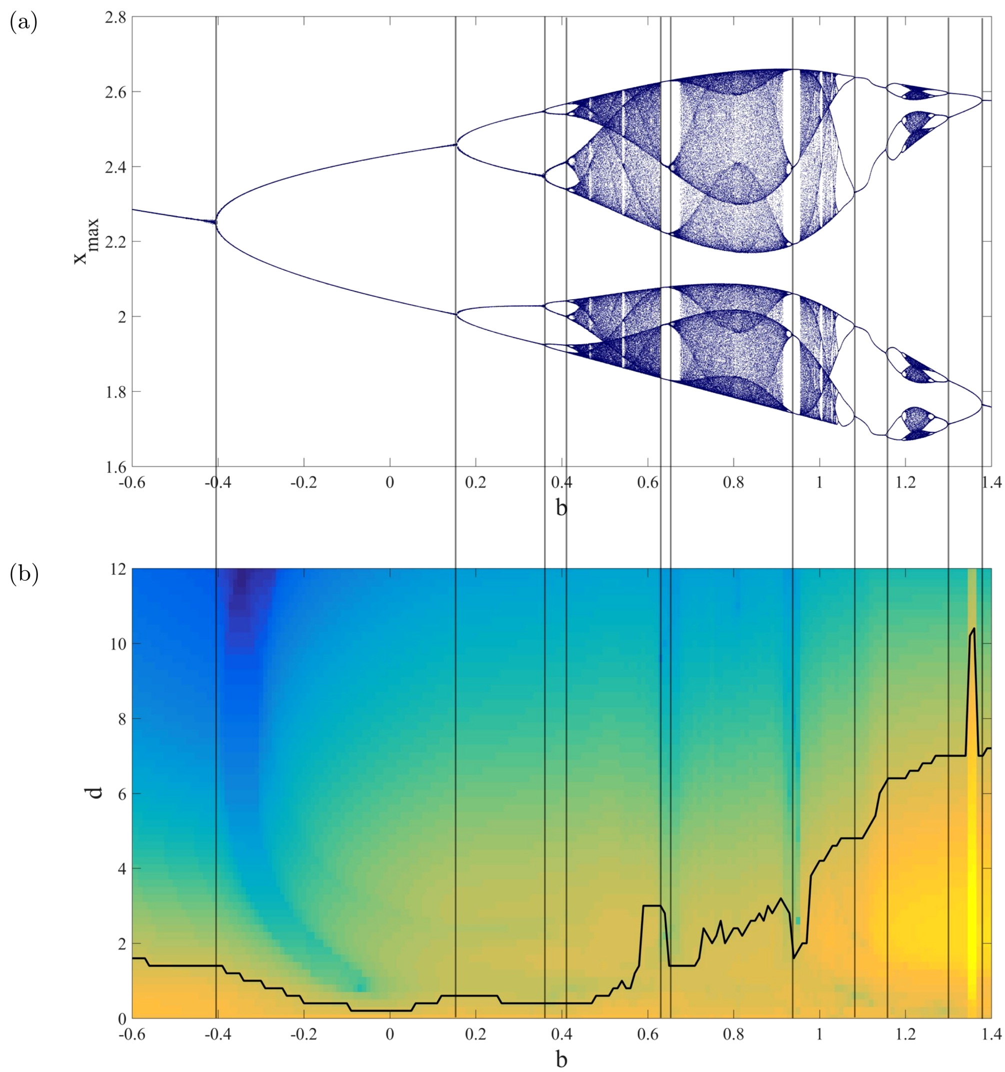

Fig. 4. (a) The bifurcation diagram of fourth system [Eq. (9)] according to parameter b. (b) The MSF diagram, in the (b, d) plane. The black curve in (b) shows where MSF equals to zero. The vertical lines connect the bifurcation points to their corresponding points in the MSF diagram.

where $a = -6$ and b is regarded as a bifurcation parameter in the interval $b = [-0.6, 1.4]$.

The variational equation of the system is designed in Eq. (10):

$$\begin{aligned}
\dot{\xi}_1 &= -d\xi_1 + 2y\xi_2, \\
\dot{\xi}_2 &= 2(x\xi_1 + y\xi_2 - z\xi_3), \\
\dot{\xi}_3 &= 2(-x\xi_1 + y\xi_2).
\end{aligned} \tag{10}$$

Figure 4(a) illustrates the bifurcation diagram of the system defined in Eq. (9) with respect to changing parameter a. The bifurcation diagram is calculated by the initial conditions with backward continuation. The diagram demonstrates period-2 oscillations followed by a period-doubling route to chaos including periodic windows. Figure 4(b), exhibits the MSF diagram of the system term according to changing parameter b as a bifurcation parameter and d as the coupling strength. By growing the bifurcation parameter, the overall trend of the black curve is upward. It means that generally by increasing the bifurcation parameter, the network is synchronized with larger value of d. In contrast to the previous systems, it seems that the variations in the black curve have some relation with the system bifurcation. For example, at $d = 0.63$ when the system bifurcates from chaotic to periodic, the black curve stops decreasing. Furthermore, it is observed that when the system period is changed, there is some variation in the MSF curve. But the variations are not general and therefore no particular relation can be inferred.

4. Conclusion

Master stability function (MSF) is a method used to find the necessary conditions for the synchronization of a network. The previous studies show that the MSF threshold, at which the network becomes synchronized, is variant for different modes of a system (e.g. periodic or chaotic mode). The aim of this paper was to find if there is any relationship between the bifurcation in a system and the MSF solution. To achieve this aim, we investigated four different systems by plotting the bifurcation diagram and calculating the MSF according to the bifurcation parameter. Then the obtained MSF diagrams were compared with the bifurcations of the system. It was observed that the MSF solution has some variations by changing the bifurcation parameter. But the variations are not related to the dynamical transitions of the systems.

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