Journal of Computational Science 68 (2023) 101997

Contents lists available at ScienceDirect

Journal of Computational Science

journal homepage: www.elsevier.com/locate/jocs

Machine learning driven extended matrix norm method for the solution of large-scale zero-sum matrix games

Burhaneddin İzgi [a,c,*], Murat Özkaya [a,d], Nazım Kemal Üre [b,c], Matjaž Perc [e,f,g,h,i]

[a] Istanbul Technical University, Department of Mathematics, Maslak, Istanbul, 34469, Turkey
[b] Istanbul Technical University, Department of Artificial Intelligence and Data Eng., Maslak, Istanbul, Turkey
[c] Artificial Intelligence and Data Science Research Center, Istanbul Technical University, Istanbul, Turkey
[d] Çanakkale Onsekiz Mart University, Faculty of Political Sciences, Business Department, Çanakkale, Turkey
[e] Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia
[f] Complexity Science Hub Vienna, Josefstädterstraße 39, 1080 Vienna, Austria
[g] Department of Medical Research, China Medical University Hospital, China Medical University, 404332 Taichung, Taiwan
[h] Alma Mater Europaea, Slovenska ulica 17, 2000 Maribor, Slovenia
[i] Department of Physics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul, Republic of Korea

ARTICLE INFO

Keywords:
Machine learning
EMN method
Large-scale games
Zero-sum games
Approximated solutions

ABSTRACT

In this paper, we develop a novel machine learning-driven framework for solving large-scale zero-sum matrix games by exploiting patterns discovered from the offline extended matrix norm method. Modern game theoretic tools such as the extended matrix norm method allow rapid estimation of the game values for small-scale zero-sum games by computing norms of the payoff matrix. However, as the number of strategies in the game increases, obtaining an accurate value estimation through the extended matrix norm method becomes more difficult. In this work, we propose a novel neural network architecture for large-scale zero-sum matrix games, which takes the estimations of the extended matrix norm method and payoff matrix as inputs, and provides a rapid estimation of the game value as the output. The proposed architecture is trained over various random zero-sum games of different dimensions. Results show that the developed framework can obtain accurate value predictions, with a less than 10% absolute relative error, for games with up to 50 strategies. Also of note, after the network is trained, solution predictions can be obtained in real-time, which makes the proposed method particularly useful for real-world applications.

1. Introduction

Game theory is a branch of science that consider conflict situations from the perspective of mathematical approaches. The game theory first appears during World War II as the result of the application of mathematical approaches to some military problems [1]. Von Neumann and Morgenstern are the first scientists formally presented the fundamentals of the game theory in 1944 by the book "Theory of Games and Economic Behavior" [2]. In 1950, Nash presented the existence of the equilibrium point for every game and he won the Nobel prize for this study in 1994 [3]. Aumann and Schelling were awarded Nobel Prize as well, in 2005, due to their contribution to the understanding of competition and cooperation in game theory [4]. Then, the game theory started to take the attention of the researchers.

In the course of time, the game theory developed theoretically and found various application areas in real-life problems. Game theory has a wide range of use such as in economics, international relations, distribution of resources, military decision, and so on. For example, in 1987, Kennedy investigated the fishery competition between Australia and Japan with the help of game theory [5]. In 1991, Lemaire studied the basics of cooperative game theory in his paper and presented some insurance applications of these types of games [6]. In 1996, Yeung presented a differential game model for the interchangeable grocery product [7]. In 1999, Singh investigated the electric power allocation by using game theory [8]. In 2000, Finus investigated international environmental problems and presented an application considering Kyto Protocol [9]. Sandler, in 2003, used the game theory to provide some policy insights for the terrorism problem [10]. Raquel et al. in 2007 applied game theory to the conflict of groundwater in Mexico [11]. In 2008, Hennet and Arda used the game theory to evaluate the efficiency of different types of contracts between the industrial partners of a supply chain [12]. In 2010, Bailey et al. studied game theory in terms of fishery and they concluded that the game theory provides insight

* Corresponding author at: Istanbul Technical University, Department of Mathematics, Maslak, Istanbul, 34469, Turkey.
E-mail address: bizgi@itu.edu.tr (B. İzgi).

https://doi.org/10.1016/j.jocs.2023.101997
Received 13 January 2023; Received in revised form 15 February 2023; Accepted 6 March 2023
Available online 15 March 2023

into achieving cooperative fishery management [13]. Soriano, in 2013, presented a brief literature review that considers the application of game theory in engineering problems [14]. Do et al. in 2017, presented the game theoretical methods used for privacy problems and cyber security [15].

In 2018, İzgi and Özkaya presented a new numerical method, which is called the matrix norm method (MN method), based on the matrix norms of the payoff matrix, that provides an approximate solution to a matrix game without solving any equations. In addition to this, they proposed a pathway for the creation of a matrix game [16]. In 2018, Özkaya studied the MN method and its application comprehensively [17]. Wang et al. improved an extended emotional model for voluntary prisoner's dilemma. They aim to show that the players do not simply imitate pure strategies, but also imitate the emotional profiles of one another instead [18]. In 2020, İzgi and Özkaya presented the necessity of agricultural insurance by using game theory [19]. In the same year, Babajanyan et al. applied the cooperative game theory to physical systems for searching an equilibrium point for the thermalization process by considering the entropy and negative energy as the utilities of two different players of the model [20]. In 2021, Özkaya and İzgi modeled an international crisis by using game theoretical tools [21]. In the same year, Li analyzed the behavior of people wearing masks by creating a game model [22]. Additionally, Özkaya and İzgi investigated the effects of the individuals' quarantine behavior on the infection risks by developing game models with the real data of three different stages of the pandemic [23]. Dhakal et al. developed an evolutionary game model that investigates the effects of trust in social and physical groups on cooperation and migration decisions. The authors claimed that their work that analyzed the concepts of tags on trust and migration by the evolutionary game theoretical approach is one of the first studies in the literature [24]. Glaubitz and Fu studied social distancing behavior by using the evolutionary game theory model in the epidemiological process in the same year [25]. Also, Tripp et al. studied a new evolutionary game theoretic framework modeling the behavior and evolution of systems of coupled oscillators [26]. Han et al. stated that it is beneficial to use multi-agent systems with the help of evolutionary game theory for improving the understanding of collective behaviors [27]. Chen and Fu analyzed the fairness and extortion by the game theoretical tools [28]. In 2023, İzgi et al. presented the extended version of the MN method, which is called the extended matrix norm (EMN) method. They refined the boundaries of the game value, and the boundaries for the extrema of the strategy set are also improved, indirectly. Additionally, they demonstrated the convergence of the MN/EMN methods [29].

In the last decades, game theory meets with artificial intelligence (AI), especially reinforcement learning, and the number of studies considering the combination of these subjects are increased. Tuyls and Nowe presented a survey that considers the application of reinforcement learning on the evolutionary game theory in terms of multi-agent systems in 2005 [30]. Nanduri and Das in 2007 gave a non-zero stochastic game model and RL-based solution framework that allows assessment of market power in day-ahead markets [31]. In 2010, Sharma and Gopal proposed a new approach that searches for synergizing broad areas of RL and Game theory [32]. Nowe et al. in 2012 presented the basic learning framework in terms of economic and game theory and demonstrated the complexity of these systems. They also described some algorithms for multi-agent reinforcement learning research [33]. Madani and Hooshyar, in 2014, presented a new methodology combining game theory and RL and use this method to develop optimal policies for multi-operator reservoir systems [34]. In 2015, Xiao et al. studied jamming games underwater and they suggested a power control strategy by using Q-Learning [35]. In 2018, Pham et al. studied cooperative and distributed reinforcement learning and game theoretic approaches together for the field coverage of drones [36]. In 2021, Albaba and Yıldız proposed a modeling framework for behavioral predictions of drivers in highway driving scenarios by using reinforcement learning and behavioral game theory [37]. Yazidi et al. provided a solution to stochastic two-person zero games with incomplete information by using learning automata [38]. Özdağlar et al. proposed a stochastic game model for multi-agent learning in dynamical systems, especially the agents play without coordination and are myopic [39]. In 2022, Wu and Lisser used the Dynamical Neural Network approach to investigate the saddle point of stochastic two-player zero-sum games [40]. Agarwal et al. studied the mean-field equilibrium in the stochastic game by using reinforcement learning [41]. İzgi et al. used the MN method and AI together, and developed an AI-supported MN method, which provides more accurate game value for matrix games [42]. The number of these kinds of studies has increased in the literature over time.

Machine learning (ML) methods, especially neural networks [43, 44] have gained significant attraction in recent years. In particular, researchers have used neural networks to replicate/clone existing optimization and search algorithms, such as branch and bound [45], to improve their solution time and performance. The main motivation behind cloning such algorithms is the fact while running the original algorithm on large-scale problems might be very time-consuming, inferring a neural network trained on the sample solutions of the original problem is usually much more time efficient. For example, in [46], authors use dimensional reducing techniques to improve the solution time of their game theoretic algorithm, since the original algorithm does not scale well to large-scale games, especially large-scale zero-sum (LS-ZS) games. That being said, there is a limited amount of previous work in using neural networks to improve solution times for game theoretic problems [47].

In this work, we propose a novel methodology for improving the scalability of the EMN method to LS-ZS games by training a neural network that exploits solution patterns discovered by EMN. The contributions of the developed method can be summarized as follows:

- We develop a machine learning model that can predict game values based on past solutions. The model leverages inputs generated with the EMN method, which solves games using matrix norms.
- We obtain value predictions with <10% absolute relative error, for games with up to 50 strategies. Moreover, predictions are obtained in real-time, which is particularly useful for real-world applications.

The remainder of the paper is organized as follows: In Section 2, we briefly present the EMN method and relevant theorems. In Section 3, we demonstrate the machine learning-based solution prediction system including the EMN method for LS-ZS matrix games. In Section 4, we present some applications of the prediction system for small, medium, and large-scale matrix games. The last section concludes the paper.

2. EMN method

In the literature, the solution of matrix games requires linear programming methods and other methods or package programs. However, MN and EMN methods do not need any solution of linear systems or such systems even for LS-ZS matrix games. In this section, we present the fundamental definition and theorems of the EMN method. We begin with the definition of a row-wise/column-wise induced matrix. Next, we present Advanced Main Theorem (AMT), which states the bounds for the game value, and also state the Refined Main Theorem (RMT) which provides an approach to obtain better bounds for the game value. Then, we state a theorem giving the upper and lower boundaries for the extrema of the strategy set. Additionally, we demonstrate the usage of the EMN method step by step for small-scale zero-sum (SS-ZS) matrix games. However, the EMN method can be used for LS-ZS matrix games as well with some difficulties such as the distribution of the elements of the strategy set. The methodology for the solution to the difficulty is investigated in the following section as the main purpose of the study.

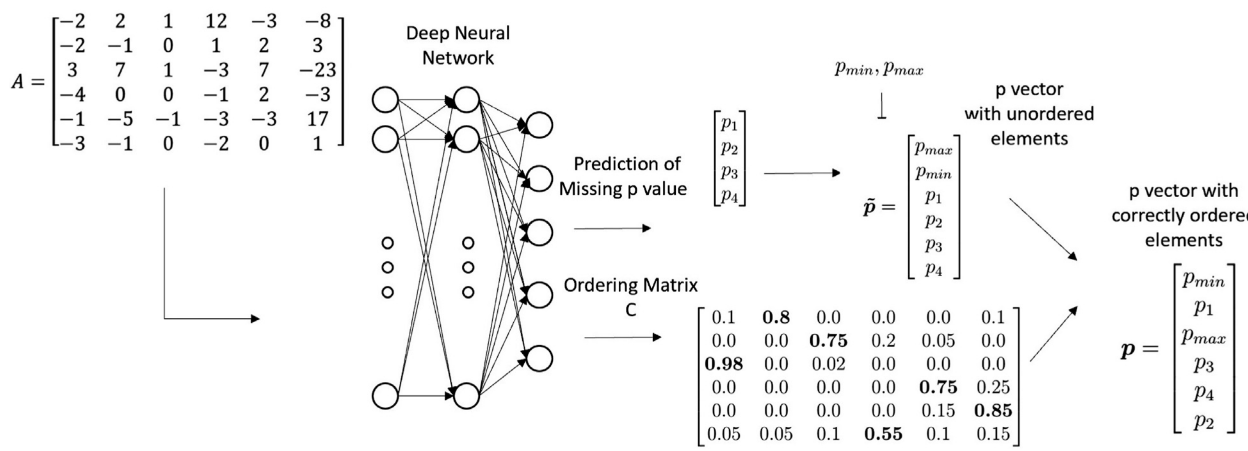


Fig. 1. Neural Network Architecture for Prediction of Missing p values. The payoff matrix A (provided by the problem description) and $p_{\max}, p_{\min}$ values (provided by the EMN algorithm) are inputs to the neural network, which processes these inputs through several convolutional and fully connected layers to predict missing values of the p vector along with soft ranking of the order of its elements.

Definition 2.1 (*[16,29]*). Let $A \in R^{m\times n}$ be a real-valued matrix, and let $\|A\|_\infty$ be the sum of absolute values of the hth row's entries, then the matrix $B \in R^{(m-1)\times n}$ is obtained by deleting hth row of the matrix A is called a row-wise induced matrix of A. Similarly, let A$\in R^{m\times n}$ be a real-valued matrix, and let $\|A\|_1$ be the sum of absolute values of the sth column's entries, then the matrix $B \in R^{m\times(n-1)}$ is obtained by deleting sth column of the matrix A is called a column-wise induced matrix of A.

Theorem 2.2 (*Advanced Main Theorem [29]*). *Let A be a m×n payoff matrix and v be the game value for a two-person zero-sum game, and define $\|A\|_{\bar{1}} = \min_j \sum_i |a_{ij}|$, $\|A\|_{\overline{\infty}} = \min_i \sum_j |a_{ij}|$. Then,*

if $|v| \geq 1, then\ L_v \leq |v| \leq U_v$

and

if $|v| \leq 1, and\ |v| \neq 0, then\ (U_v)^{-1} \leq |v| \leq (L_v)^{-1}$

where $L_v = \max\left\{ \frac{\|B\|_\infty}{\|A\|_\infty}, \frac{\|\hat{B}\|_1}{\|A\|_1} \right\}$ *and* $U_v = \min\left\{ \|A\|_{\bar{1}}, \|A\|_{\overline{\infty}} \right\}$, *and B and $\hat{B}$ are the row-wise and column-wise induced matrices of A, respectively.*

Theorem 2.3 (*Refined Main Theorem [29]*). *Let $A \in \mathbb{R}^{m\times n}$ be the payoff matrix with positive entries of a two-person zero-sum game. Then, $p_{min}\|A\|_1 \leq v \leq p_{max}\|A\|_{\bar{1}}$ holds where $\|A\|_{\bar{1}} = \min_j \sum_i |a_{ij}|$, and p_{max} and p_{min} are the greatest and smallest elements in the mixed strategy set, respectively.*

By using AMT, RMT, and Proposition 2.7, (if required), in [16] we can find an approximate game value for a matrix game. In addition to these, İzgi and Özkaya presented lower and upper bounds for the maximum and minimum elements of the mixed strategy set in [16] by the following theorem.

Theorem 2.4 (*[16,29]*). *Let $A \in R^{m\times n}$ be the payoff matrix with positive entries. Then, the boundaries for p_{max} and p_{min} which are the greatest and the smallest elements of the mixed strategy set, respectively, are as follows,*

$$p_{max} \geq L\ where\ L = max\left\{ \frac{1 - \frac{v}{\|A\|_1}}{m-1}, \frac{v}{\|B\|_1} \right\}$$

and

$$p_{min} \leq U\ where\ U = min\left\{ \frac{1 - \frac{v}{\|B\|_1}}{m-1}, \frac{v}{\|A\|_1} \right\}$$

where B is the column-wise induced matrix of A.

In order to use the above theorems, we need to have a payoff matrix with positive entries. However, we know that every matrix game can be converted to a matrix game with nonzero entries by Proposition 2.7. in [16]. Therefore, we can use the given theorem for all types of matrix games.

Briefly, the EMN method can be applied to any size of matrix games by pursuing the following steps:

1. Use AMT and obtain an interval containing the game value.
2. Choose the midpoint of the interval as the dummy game value for simplicity.
3. Use Theorem 2.4 and find an upper and lower bound for the minimum (p_{min}) and maximum (p_{max}) elements of the mixed strategy set, respectively.
4. Use RMT to update a new interval for the game value and select a dummy game value as in Step 2.
5. Repeat Step 3 to obtain better bounds for p_{max} and p_{min}.
6. Create the proper mixed strategy set by considering the basic principles of the probability theory and the bounds.
7. Evaluate the approximate game value, v_{app}.

These steps represent a general flowchart of the EMN method to matrix games that are used throughout the analyses being considered in this paper.

3. Machine learning based solution prediction system for EMN to LS-ZS matrix games

In general, it is hard to obtain the analytic solutions of the LS-ZS matrix games rather than SS-ZS matrix games due to the natural structure of the matrix games. Therefore, LS-ZS matrix games are not preferred to model most of the problems directly although they could reflect or realize the real-life problem better than the SS-ZS games. Because of this difficulty, scientists started working on dimension reduction or other numerical techniques for the LS-ZS matrix games to ease the problem in the literature. For instance, Li et al. tries to solve relatively large-scale matrix games by using the dimension reduction technique to overcome the issue. In this context, they modeled an air vehicle combat as 16×12 matrix and analyzed it by using the dimensionality reduction method. They compared their results to different methodologies [46].

On the other hand, it is clear that the EMN method seems useful for zero-sum (and also non-zero sum) matrix games [16,29] but it is not very practical for LS-ZS matrix games due to the difficulty of the proper distribution of the elements of the mixed strategy set. In

Table 1
Value prediction error of the developed models across different game sizes and the number of training samples. It can be observed that as the size of the games increases, the average prediction error on both training and test sets get worse, since the model needs more samples to fit to the data on higher dimensional problems. On the other hand, for a fixed game size, increasing the number of samples improves the error rate on the test set significantly, whereas the error rate on the training set only increases slightly. These results reinforce that there is indeed a learnable pattern between the structure of the payoff matrix and the game value, which can be discovered by using machine learning algorithms with appropriate architectures.

Number of games used for model development	Value prediction error on training games (80% of all games)			Value prediction error on test games (20% of all Games		
Size of games	5 × 5	10 × 10	50 × 50	5 × 5	10 × 10	50 × 50
500	4.20%	4.90%	7.30%	33.20%	36.30%	40.30%
1000	4.70%	5.70%	6.20%	10.10%	12.10%	32.10%
2000	5.10%	6.10%	10.10%	6.20%	8.10%	28.10%
10000	5.20%	8.20%	11.90%	5.60%	9.10%	14.70%
20000	5.30%	9.30%	13.10%	5.50%	9.70%	14.20%
50000	5.30%	7.20%	10.30%	5.50%	8.40%	11.50%
100000	4.40%	6.40%	8.40%	5.20%	7.90%	10.20%
200000	4.60%	5.60%	8.20%	5.00%	6.10%	9.90%
500000	4.80%	5.80%	8.30%	4.90%	5.50%	9.90%

general aspects, it is obvious that when the matrix size increases, the distribution of the elements of the strategy set becomes complicated and requires more calculation and fortune. Therefore, we aim to use Artificial Intelligence (AI) in order to annihilate the difficulty so that we combine the EMN method and AI.

First of all, we develop a neural network that predicts the relatively optimal distribution of the elements of the mixed strategy set that is obtained by the EMN method of the game in this section. Then, for the predictions, we provide only the extrema of the strategy set to the machine to distribute the elements in the set properly. In other words, we use the neural network to seek a suitable strategy set by using the obtained information from the EMN method.

Here, our proposed architecture is displayed in Fig. 1. A forward pass through the network works as follows; first, the payoff matrix of the game is provided as an input. The network processes the payoff matrix using convolutions layers and projects it into a latent space vector. Next, this latent vector is concatenated with the p_{min} and p_{max} values predicted with the EMN algorithm, and the network passes this information from several dense and convolutions layers to predict two outputs. The first output is the missing p values, and the second output is a probability matrix that predicts the probability of the location of each of the missing values. By taking the maximum column element of this matrix, it is possible to reconstruct an approximate p vector, which can then be used to estimate an approximate value of the game.

4. Applications and results

In this section, we present the corresponding algorithm for the proposed method. First, we provide the matrix game creation algorithm. Next, we give the algorithm for the proposed neural network architecture. After that, we briefly mention our database. As a result of this section, we present in detail our results for the solution of various dimensions of the matrix game and the performance of the neural network architecture. We begin with presenting Algorithm 1 which is developed to generate random matrix games in different sizes and solve them. In addition to these, the algorithm also includes the applications of the EMN method for determining the extrema of the strategy sets for the corresponding matrix games in order to obtain a dataset to train the neural network model.

By conducting Algorithm 1, we generate and consider three different sizes of matrix game, 5 × 5 (small scale), 10 × 10 (medium scale), and 50 × 50 (large scale). For each size, we generate up to 5×10^5 random payoff matrices. We reserve 80% of the sampled data to train the neural network model and use the 20% of data to check the model's performance on unseen data. Moreover, architectural details of the neural network portrayed in Fig. 1 is provided as follows:

Algorithm 1 Matrix Game Data Set Generator and conducting of the EMN method

for $i = 1, 2, \dots, N$ **do**
  Generate $A = randi([x\ y], m, n)$
  Solve the matrix A by $game\_solve(A)$ function and store in an array
  Evaluate the related norms and store in an array
  while $tol > 0.01$
  Compute v_lower, v_upper, and store the v_dummy in the array $dummystore$
  Calculate p_min and p_max, and store in the array $strategy\ set$
  $tol = abs(dummystore(end) - dummystore(end - 1))$
  end while
end for

where N represents the number of the game desired, x and y are positive integers with $x < y$, m and n denotes the size of the matrix.

- The $N \times N$ payoff matrix is first normalized so that the training data set has zero mean and unit variance. Next, 3 stacked convolutional layers are applied on the matrix data, where each layer has 3 × 3 filters and ReLu activation function. The number of filters in each layer are $64, 128, 256$ correspondingly. MaxPooling is applied in between layers 1 and 2. After layer 3, a flattening operation is applied to transform the output of layer 3 into a 1D vector.
- The output of the convolutional layers is passed through 2 dense layers, with output sizes 64, 32. These layers also have ReLU activation.
- Next, the output of dense layers is fed to two different heads, one for predicting the missing values of the p vector, and the other for estimating the relative order of the elements of the p vector. The first head is a dense layer with $N - 2$ output size and linear activation. The second head is a dense layer with $N \times N$ output size, where each output column is passed through a softmax activation.
- In the final layer predicted $N-2$ elements are fused with p_{min}, p_{max} from the data set (note that these values are written to the database by the EMN algorithm), and argmax layer is applied upon the predicted order matrix to re-order the elements of the fused p vector. The predicted p vector can be used for estimating the value of the game. Finally, loss per sample is computed by calculating the mean square error between the predicted game value and the ground truth game value.

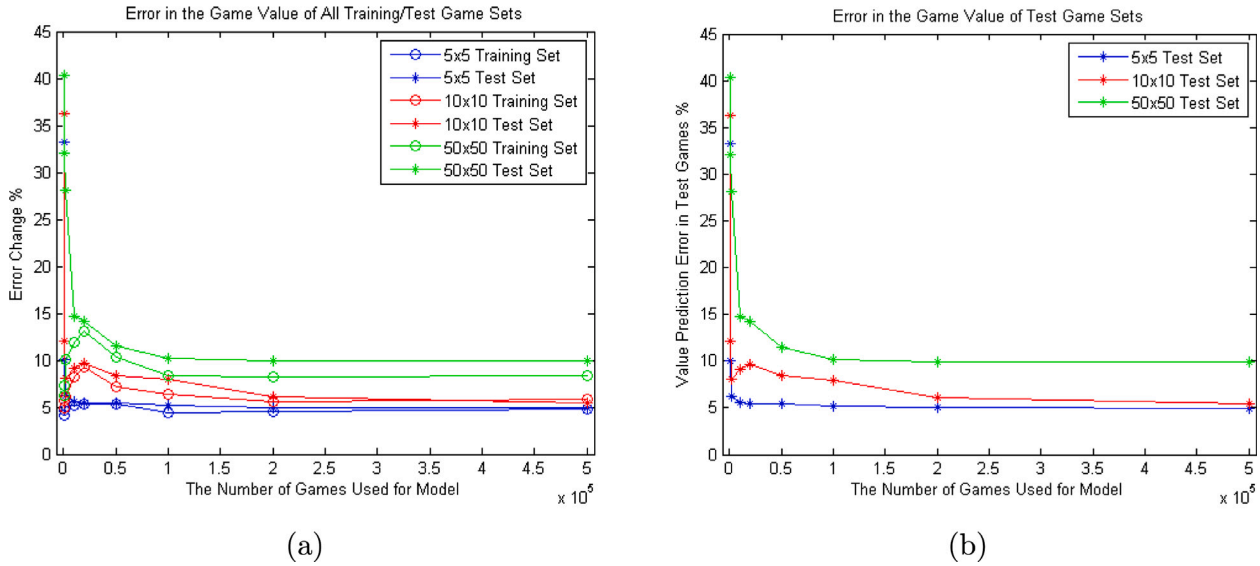


Fig. 2. Changes in the value prediction error vs. number of samples used in model development across training and test sets. It can be seen that error on the test set can be significantly reduced by increasing the number of samples (i.e. number of randomly generated payoff matrices), and the error reduction rate depends strongly on the size of the game. Larger games require more samples to close the gap between training and test error. This result, also known as the curse of dimensionality, is a common occurrence in the development of machine learning systems.

For training the network, we used the Adam optimizer with an initial learning rate 10^{-3} and batch size of 32, and a validation split of 10%. Each training instance was trained for 1000 epochs, however, learning is terminated when the validation loss did not improve for 20 epochs.

Table 1 shows the algorithms' game value prediction performance across different game and dataset sizes. It can be seen that the neural network model's generalization performance improves greatly as the number of training instances increase. In particular, for small and medium instances, the network predicts the value of the game within 5%–6% of the actual game value. For the LS-ZS matrix games, the average error is around 10%, which is still a remarkable performance given the size of these games.

Figs. 2(a) and 2(b) proved an overview of how percentage error changes as a function of the number of training samples, for different game sizes. It can be observed from these figures that all error rates decay consistently as the number of samples is increased, which supports our hypothesis that a strong relationship exists between the actual game value, pay-off matrix, and the p values computed by the EMN algorithm, which can be discovered by training machine learning algorithms from past solutions. Secondly, we can see that the decay rate of the error depends on the game size, as expected the bigger the size of the game, it takes more samples to bring down the error rate to the desired level.

Note that one of the main contributions of our work is, once the neural network is trained for a specific problem size, the solution time becomes constant in the sense that neural network (coupled with EMN algorithm) will always predict the solution to the problem using the same amount of computation. This is in contrast with some of the existing works for solving large-scale games, such as [46], where the solution time can change significantly based on the structure of the problem.

5. Conclusions

In this work, we first present the EMN method and the corresponding theorems in detail. We also provide the algorithm for the creation process for any size of matrix games and the general procedure for the EMN method. Then, we develop a machine learning model that can help scale up the EMN method to obtain approximate values for LS-ZS matrix games. Our model consists of a neural network that is trained on a large number of random zero-sum matrix games and approximate solutions offered by the EMN algorithm. We show that the proposed methodology can obtain high-quality approximate value prediction for LS-ZS matrix games. Finally, we present the comparison of the results obtained for 5×5, 10×10, and 50×50 by tables and graphs. The results indicate that we can possibly obtain better approximations of the game values by training the neural network with the larger training datasets as is expected in general. One of the main advantages offered by our methodology is, after the training phase, solutions to matrix games are obtained in constant time by performing inference on the neural network, which makes our methodology applicable to real-world problems with large matrices. For future work, we are planning to extend our methodology to stochastic games and sequential games.

CRediT authorship contribution statement

Burhaneddin İzgi: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Supervision, Project administration, Funding acquisition. **Murat Özkaya:** Data curation, Writing – original draft, Writing – review & editing, Visualization, Investigation. **Nazım Kemal Üre:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Visualization, Writing – original draft, Writing – review & editing. **Matjaž Perc:** Formal analysis, Visualization, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

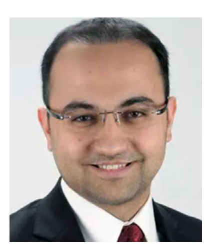
No data was used for the research described in the article.

Acknowledgments

This work is supported by the Scientific and Technological Research Council of Turkey(in Turkish: TÜBİTAK) under grant agreement 121E394. M.P. is supported by the Slovenian Research Agency (Grant Nos. P1-0403 and J1-2457). B.İ. and N.K.Ü. are supported by the Scientific Research Project Unit (BAP) of Istanbul Technical University (project no. MOA-2019-42321). The authors would like to thank the anonymous referees and the editor for their valuable suggestions and comments that helped to improve the content of the article.

References

[1] M. Shubik, Game Theory and Related Approaches To Social Behavior: Selections, John Wiley & Sons, 1964.
[2] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944.
[3] J.F. Nash, The bargaining problem, Econometrica 18 (2) (1950) 155–162.
[4] R. Aumann, T. Schelling, Contributions to game theory: Analysis of conflict and cooperation, Nobel Prize Econ. Documents 2005 (2005) 1.
[5] J.O.S. Kennedy, A computable game theoretic approach to modelling competitive fishing, Mar. Resour. Econ. 4 (1) (1987) 1–14.
[6] J. Lemaire, Cooperative game theory and its insurance applications, Astin Bull. 21 (1) (1991) 17–40.
[7] D.W.K. Yeung, A differential game model of a market of substitutable products, European J. Oper. Res. 90 (3) (1996) 599–608.
[8] H. Singh, Introduction to game theory and its application in electric power markets, IEEE Comput. Appl. Power 12 (4) (1999) 18–20.
[9] M. Finus, Game theory and international environmental cooperation: A survey with an application to the Kyoto-Protocol, Nota di Lavoro, 86, Fondazione Eni Enrico Mattei, Milano, 2000.
[10] T. Sandler, D.G. Arce, Terrorism & game theory, Simul. Gaming 34 (3) (2003) 319–337.
[11] S. Raquel, S. Ferenc, C. Emery Jr., R. Abraham, Application of game theory for a groundwater conflict in Mexico, J. Environ. Manag. 84 (4) (2007) 560–571.
[12] J.C. Hennet, Y. Arda, Supply chain coordination: A game-theory approach, Eng. Appl. Artif. Intell. 21 (3) (2008) 399–405.
[13] M. Bailey, U.R. Sumaila, M. Lindroos, Application of game theory to fisheries over three decades, Fish. Res. 102 (1–2) (2010) 1–8.
[14] J. Sanchez-Soriano, An overview on game theory applications to engineering, Int. Game Theory Rev. 15 (3) (2013) 1340019.
[15] C.T. Do, N.H. Tran, C. Hong, C.A. Kamhoua, K.A. Kwiat, E. Blasch, S. Ren, N. Pissinou, S.S. Iyengar, Game theory for cyber security and privacy, ACM Comput. Surv. 50 (2) (2017) 1–37.
[16] B. İzgi, M. Özkaya, A new perspective to the solution and creation of zero sum matrix game with matrix norms, Appl. Math. Comput. 341 (2019) 148–159.
[17] M. Özkaya, The roles of matrix norms in the game theory, (M.Sc. Thesis), Istanbul Technical University, Turkey, 2018.
[18] L. Wang, S. Ye, K.H. Cheong, W. Bao, N. Xie, The role of emotions in spatial prisoner's dilemma game with voluntary participation, Phys. A 490 (2018) 1396–1407.
[19] B. İzgi, M. Özkaya, The demonstration of the necessity of agriculture insurance by the game theory: Matrix norm approach, AKU J. Sci. Eng. 20 (5) (2020) 824–831, (in Turkish).
[20] S.G. Babajanyan, A.E. Allahverdyan, K.H. Cheong, Energy and entropy: Path from game theory to statistical mechanics, Phys. Rev. Res. 2 (4) (2020) 043055.
[21] M. Özkaya, B. İzgi, Mathematical modeling of an international crisis with game theory, BEU J. Sci. 10 (4) (2021) 1334–1341, (in Turkish).
[22] Q. Li, Application of game theory on people dealing with masks, in: 6th International Conference on Social Sciences and Economic Development, Atlantis Press, 2021, pp. 1048–1051.
[23] M. Özkaya, B. Izgi, Effects of the quarantine on the individuals' risk of Covid-19 infection: Game theoretical approach, Alex. Eng. J. 60 (4) (2021) 4157–4165.
[24] S. Dhakal, R. Chiong, M. Chica, T.A. Han, Evolution of cooperation and trust in an N-player social dilemma game with tags for migration decisions, R. Soc. Open Sci. 9 (5) (2022) 9212000212000.
[25] A. Glaubitz, F. Fu, Oscillatory dynamics in the dilemma of social distancing, Proc. R. Soc. A. 476 (2022) 20200686.
[26] E.A. Tripp, F. Fu, S.D. Pauls, Evolutionary Kuramoto dynamics, Proc. R. Soc. B. 289 (2022) 20220999.
[27] T.A. Han, S.V. Albrecht, M. Woolridge, Emergent behaviours in multi-agent systems with evolutionary game theory, AI Commun. 35 (4) (2022) 327–337.
[28] X. Chen, F. Fu, Outlearning extortioners by fair-minded unbending strategies, 2022, arXiv preprint, arXiv:2201.04198.
[29] B. İzgi, M. Özkaya, N.K. Üre, M. Perc, Extended matrix norm method: Applications to bimatrix games and convergence results, Appl. Math. Comput. 438 (2023) 127553.
[30] K. Tuyls, A. Nowé, Evolutionary game theory and multi-agent reinforcement learning, Knowl. Eng. Rev. 20 (1) (2005) 63–90.
[31] V. Nanduri, T.K. Das, A reinforcement learning model to assess market power under auction-based energy pricing, IEEE Trans. Power Syst. 22 (1) (2007) 85–95.
[32] R. Sharma, M. Gopal, Synergizing reinforcement learning and game theory-A new direction for control, Appl. Soft Comput. 10 (3) (2010) 675–688.
[33] A. Nowé, P. Vrancx, Y.M.D. Hauwere, Game theory and multi-agent reinforcement learning, in: Reinforcement Learning, Springer, Berlin, Heidelberg, 2012, pp. 441–470.
[34] K. Madani, M. Hooshyar, A game theory–reinforcement learning (GT–RL) method to develop optimal operation policies for multi-operator reservoir systems, J. Hydrol. 519 (A) (2014) 732–742.
[35] L. Xiao, Q. Li, T. Chen, E. Cheng, H. Dai, Jamming games in underwater sensor networks with reinforcement learning, in: 2015 IEEE Global Communications Conference, 2015, pp. 1-6.
[36] H.X. Pham, H.M. La, D. Feil-Seifer, A. Nefian, Cooperative and distributed reinforcement learning of drones for field coverage, 2018, arXiv preprint, arXiv:1803.07250.
[37] B.M. Albaba, Y. Yildiz, Driver modeling through deep reinforcement learning and behavioral game theory, IEEE Trans. Control Syst. Technol. 30 (2) (2021) 885–892.
[38] A. Yazidi, D. Silvestre, B.J. Oommen, Solving two-person zero-sum stochastic games with incomplete information using learning automata with artificial barriers, IEEE Trans. Neural Netw. Learn. Syst. (2021) 1–12.
[39] A. Ozdaglar, M.O. Sayin, K. Zhang, Independent learning in stochastic games, 2021, arXiv preprint, arXiv:2111.11743.
[40] D. Wu, A. Lisser, A dynamical neural network approach for solving stochastic two-player zero-sum games, Neural Netw. 152 (2022) 140–149.
[41] M. Agarwal, V. Aggarwal, A. Ghosh, N. Tiwari, Reinforcement learning for mean-field game, Algorithms 15 (3) (2022) 73.
[42] B. İzgi, M. Özkaya, N.K. Üre, Application of matrix norm method with artificial intelligence, in: Proceedings of 2nd International Congress on Modern Sciences, IKSAD Publications, 2022, pp. 806–815, (in Turkish).
[43] Y. LeCun, Y. Bengio, G. Hinton, Deep learning, Nature 521 (2015) 436–444.
[44] I. Goodfellow, Y. Bengio, A. Courville, Deep Learning, MIT Press, 2016.
[45] G. Zarpellon, J. Jo, A. Lodi, Y. Bengio, Parameterizing branch-and-bound search trees to learn branching policies, in: Proceedings of the AAAI Conference on Artificial Intelligence, 35, vol. 5, 2021, pp. 3931-3939.
[46] S. Li, M. Chen, Y. Wang, Q. Wu, A fast algorithm to solve large-scale matrix games based on dimensionality reduction and its application in multiple unmanned combat air vehicles attack-defense decision-making, Inform. Sci. 594 (2022) 305–321.
[47] D. Wu, A. Lisser, A dynamical neural network approach for solving stochastic two-player zero-sum games, Neural Netw. 152 (2022) 140–149.



Burhaneddin İzgi received his Ph.D. degree in Mathematical Engineering from Istanbul Technical University (ITU) in 2015. He worked as a visiting researcher in the department of Mathematics and Statistics at California State University-Sacramento in 2009 and 2014. Dr. İzgi is currently working as an Associate Professor at ITU. He is also a member of the ITU Artificial Intelligence Research Center (ITU AI). His main research interests are related to applied mathematics, mathematical finance, behavioral classification of stochastic differential equations (SDEs), numerical methods for the solution of SDEs, game theory and its artificial intelligence applications.

Murat Özkaya received his B.Sc. degree in mathematics from Abant Izzet Baysal University in 2016, and the M.Sc. degree in mathematical engineering from Istanbul Technical **University in 2018. He** is a Ph.D. candidate in mathematical engineering at Istanbul Technical University. He currently works at Çanakkale Onsekiz Mart University as a research assistant. His research interests include game theory, decision theory and their applications.

Nazim Kemal Ure obtained his Ph.D. degree in Aerospace Engineering from Massachusetts Institute of Technology (MIT) in 2015. Dr. Ure is currently an Associate Professor at Istanbul Technical University (ITU), department of Artificial Intelligence and Data Engineering, he also serves as the Vice Director of ITU Artificial Intelligence Research Center (ITU AI). His main research interests are applications of deep learning and deep reinforcement learning for autonomous systems, large scale optimization, and development of high-performance guidance navigation and control algorithms.

Matjaž Perc is professor of physics at the University of Maribor. He is a member of Academia Europaea and the European Academy of Sciences and Arts, and among top 1% most cited physicists according to 2020 and 2021 Clarivate Analytics data. He is also the 2015 recipient of the Young Scientist Award for Socio and Econophysics from the German Physical Society, and the 2017 USERN Laureate. In 2018 he received the Zois Award, which is the highest national research award in Slovenia. In 2019 he became Fellow of the American Physical Society. Since 2021 he is Vice Dean of Natural Sciences at the European Academy of Sciences and Arts.