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Axioms of Decision Criteria for 3D Matrix Games and Their Applications

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Abstract: In this paper, we define characteristic axioms for 3D matrix games and extend the definitions of the decision criteria under uncertainty to three dimensions in order to investigate the simultaneous effect of two different states on the decision process. We first redefine the Laplace, Wald, Hurwicz, and Savage criteria in 3D. We present a new definition depending on only the ∞-norm of the 3D payoff matrix for the Laplace criterion in 3D. Then, we demonstrate that the Laplace criterion in 3D explicitly satisfies all the proposed axioms, as well as the other three criteria. Moreover, we illustrate a fundamental example for a three-dimensional matrix with 3D figures and show the usage of each criterion in detail. In the second example, we model a decision process during the COVID-19 pandemic for South Korea to show the applicability of the 3D decision criteria using real data with two different states of nature for individuals' actions for the quarantine. Additionally, we present an agricultural insurance problem and analyze the effects of the hailstorm and different speeds of wind on the harvest by the 3D criteria. To the best of our knowledge, this is the first study that brings 3D matrices in decision and game theories together.

Keywords: characteristic axioms; multi-state games; three-dimensional matrix games; game against nature; COVID-19; insurance problem

MSC: 91Axx; 91A35; 91A80; 91B06

1. Introduction

The decision-making process involves some difficulties such as choosing among a large number of criteria, and alternatives, the determination of the importance of the criteria (i.e., the weights of the criteria), collecting the related information about the problem, and developing a model for the decision-making problem [1–3]. The process also requires significant time and effort to analyze the numerous actions using systematic techniques. In particular, decision-making under uncertainty is more difficult since there is no knowledge about the player's behavior [4]. These types of games are known as the games against nature in the literature. Games against nature are the games in which the payoffs in the game are unknown and the probability with which the player will choose his/her actions is entirely unknown [5,6]. Pazek and Rozman stated how a player decides under uncertainty in [7] as follows:

1. Make a list of all possible alternatives for obtaining knowledge, experimenting, and taking action.

2. List all events that may possibly occur.
3. Arrange all relevant data and choices/assumptions.
4. Sort the effects of each choice of action on a scale.
5. Determine the probability of an uncertain event occurring.

Over time, some mathematical methods have been developed for decision-making under uncertainty. For example, Laplace, Wald, Hurwicz, and Savage presented some criteria for these types of games [8–10]. On other hand, the results of decisions not only depend on the decisions themselves, but also on some exterior factors that cannot be controlled by the player. The word "nature" in games against nature stands for the entirety of these external factors. The state of nature refers to the complete explanation of the external factors concerning all aspects of the situation [11]. Researchers have improved the methodologies and presented more complex algorithms to find an answer for such decision processes, which are known as decision criteria under uncertainty or multicriteria decision methods. Then, these methods are applied to various types of problems. For example, the criteria under uncertainty are usually applied to agricultural problems in the literature due to the unpredictable behavior of nature [7,12,13]. However, the application area of these criteria is not limited to only agricultural applications. For example, in 2006, Ballestero et al. used these criteria to analyze the best portfolio options in the Frankfurt and Vienna Stock Exchanges [14]. In 2007, Ünal and Atılgan applied the decision-making techniques to the apparel industry [15]. Galasso and Thiery, in 2009, investigated the consumer–supplier relationship using these criteria [16].

Green and Weatherhead, in 2014, studied the climate change problem with the criteria under uncertainty [17]. In 2019, Ma et al. used the multicriteria decision technique to select a portfolio fulfilling the requirement [18]. In 2020, İzgi and Özkaya considered the necessity of the agricultural insurance problem by these criteria with a method called the matrix norm approach from game theory [12]. In the same year, Yazdani et al. used the multicriteria methods to decide the optimal location of healthcare waste disposal [19]. Furthermore, in 2020, Saldanha et al. showed that multicriteria decision-making under uncertainty provides a rational solution to shell-and-tube heat problems [20]. In 2021, Ulansky and Raza proposed an approach that generalized the decision criteria under uncertainty [21]. Shmelova et al. investigated the selection of the optimal location for a remote tower center by using the criteria of Laplace, Wald, Savage, and Hurwicz [22]. Pak et al., in 2021, applied the criteria for assessing the economic efficiency of investment projects under the conditions of radical and probabilistic uncertainties [23]. Shmelova and Sechko studied the hybrid expert system for collaborative decision-making in transportation services of healthcare needs by using the decision criteria under uncertainty [24]. Wang et al. aimed to develop an indicator framework for integrated monitoring of the costing process for a patient by rehabilitation medical institutions with the help of decision criteria [25]. Balasa et al., in 2022, investigated the risk management and risk control in a wind tunnel by the criteria under uncertainty [26]. Gomes and Martins represented the details of the decision criteria under uncertainty in their book in detail and also presented some applications of the criteria [27]. Balakina et al. used the criterion under uncertainty to demonstrate how large-scale railway projects of federal significance should be computed [28]. Vdovyn et al. used Hurwicz, Bayes–Laplace, Wald, Savage, and another decision criterion to model economic systems by game theory [29]. Walker et al. studied the effects of future developments on buildings' greenhouse gas emissions by scenario-based robustness assessment with the help of decision criteria [30].

In addition to these, in this paper, we mainly focus on the methods established by Laplace, Wald, Hurwicz, and Savage and combine the 3D matrices together. All the definitions and applications above are modeled with two-dimensional matrices and related matrix operations. However, it may be more useful to model a problem with higher-dimensional matrices depending on the type of problem since those matrices may express and reflect the problem better than it is stated in two dimensions. To develop a higher-dimensional model with matrices, scientists began by defining the 3D matrices in the literature. Even though there are many different definitions of the 3D matrices, we ground ourselves on the definitions in [31]. İzgi, in 2015, explicitly defined and proved 3D matrix norms with details in his Ph.D. thesis. Additionally, the applications of the 3D matrix norms to mathematical finance were presented in [31,32]. In 2017, İzgi and Özkaya stated and proved the equivalence of the 3D matrix norms. They demonstrated an application of these norms' inequalities on some simulation results obtained by real data using the methods of mathematical finance [33]. In 2018, İzgi and Özkaya presented the fundamental definitions and properties of three-dimensional matrices [34].

Besides the numerous theoretical studies about 3D matrices in the literature, there are different applications of 3D matrices in physics, electrical engineering, mathematical finance, etc. In 1979, Cosima proved that, if each of the horizontal plane sections of the $p \times q \times r$, three-dimensional matrix A, has full-term rank, then the plane term rank of the 3D matrix A is larger than $m - \sqrt{m}$, where $m = min\{p, q, r\}$, which is used in physics [35]. In 1996, Burkard et al. investigated 3D assignment problems as considering the cost coefficients can be decomposed into three different values [36]. In 2006, Ignatova and Styczynski showed the graphical representation of the transition number matrix and the Markov matrix in 3D [37]. As seen from the studies above, three-dimensional matrices have a wide range of use in the literature.

In this paper, we specifically study the decision under uncertainty and its fundamental criteria such as those of Laplace, Wald, Hurwicz, and Savage. These criteria consist of only one state; in other words, these criteria may refer to the single-state decision criteria. We present the extension of the criteria under uncertainty to the three-dimensional cases by adding a second state of nature in the payoff matrix by considering that the decision-making process in real-life problems is usually affected by more than one factor. Therefore, we extend the definitions of the Laplace, Wald, Hurwicz, and Savage criteria to the three-dimensional case to capture and analyze the effects of the second state of nature, the multiple state decision criteria. In the literature, there are studies about the multiple criteria/multicriteria decision-making process [38–40]. However, this study improves the multiple-state decision-making process based on the decision criteria under uncertainty and 3D matrices. The number of states may be increased by the usage of higher-dimensional matrices. According to our literature review, the decision and game theories come together with the 3D matrices for the first time in this study. Thus, we aim to bring a new perspective to game theory and decision theory by these extensions, that is the multi-state decision criteria, via the usage of 3D matrices.

The remainder of the paper is organized as follows: In Section 1.1, we present some theoretical background and definitions of Laplace, Wald, Hurwicz, and Savage. In Section 2, we present the characteristic axioms of the criteria in 3D. Moreover, we define the three-dimensional decision criteria such as those of Laplace, Wald, Hurwicz, and Savage. Additionally, we demonstrate that each criterion satisfies the characteristic axioms of the criteria in 3D. In Section 3, we give a fundamental example explaining the usage of the 3D criteria. Furthermore, we present real-life problems modeled with 3D matrices and solve them by using the 3D decision criteria, explicitly. Section 4 concludes the paper.

1.1. Some Theoretical Background

In this section, we present some notions and definitions that are used as a basis throughout the study. To show the correspondence between the state of nature and actions selected by a player, we first focus on the representation of the game against nature. The consequences of actions depending on states are usually represented as in Table 1, including the outcome of the actions [6,11].

Table 1. Representation of game against nature [11].

		State of Nature			
	Values	S_1	S_2	$\dots$	S_n
Actions	A_1	V_{11}	V_{12}	$\dots$	V_{1n}
	A_2	.	.	$\dots$	.
	.	.	.	$\dots$	.
	.	.	.	$\dots$	.
	.	.	.	$\dots$	.
	A_m	V_{m1}	V_{m2}	$\dots$	V_{mn}

In decision theory, there are different types of decisions such as decision under risk, decision under certainty, decision under uncertainty, and multicriteria decision [5,41]. The types of uncertainty are called by different authors in various [42–44]. However, we can summarize them as follows [11]:

Definition 1 (First type of uncertainty)**.** *The player knows the states of nature.*

Definition 2 (Second type of uncertainty)**.** *The player knows that states of nature, but not the probabilities.*

Definition 3 (Third type of uncertainty)**.** *The player knows both the states of nature and the probabilities.*

The player decides the decision rule after stating the problems clearly and analyzing the situations. The best-known decision-making rules are those of the Laplace, Wald, Hurwicz, and Savage criteria, among others [11].

Definition 4 (Laplace criterion)**.** *Let p_{ij} denote the probabilities of the n states of nature and v_{ij} denote the outcomes by the usage of the i^{th} action chosen by the player within the j^{th} state of nature. The Laplace value, also known as the expected monetary value, E_i for the action A_i is evaluated as*

$$E_i = \sum_{j=1}^{n} p_{ij} v_{ij}.$$

Then, the Laplace criterion implies the action with a maximum E_i so that we select

$$A_{opt} = \max_{i=1\dots m} \{E_i\}.$$

In addition to this definition, Laplace defines the criterion in [8] as a maximum of

$$r_i = \frac{1}{n} \sum_{j=1}^{n} v_{ij}.$$

Definition 5 (Wald criterion)**.** *The worst potential outcome that might occur as a result of action A_i has a value for the decision-maker of*

$$W_i = \min_j \{v_{ij}\}$$

where W_i is the security level of the action A_i. Wald recommends that the player should select the action with the highest security level as:

$$W_{opt} = \max_i \{W_i\}.$$

Definition 6 (Hurwicz criterion)**.** *Hurwicz established an optimistic criterion by taking into account the best-possible result of each action. The optimism level is defined as*

$$H_i = \max_j \{v_{ij}\}$$

where H_i is the best result if action A_i is taken. Hurwicz's maximax return criterion is

$$H_{opt} = \max_i \{H_i\}.$$

Hurwicz also defined the optimism–pessimism index z with $0 \leq z \leq 1$ in 1951 by considering a balance between maximum optimism and pessimism [9]. Moreover, Hurwicz proposed the decision rule with regard to the index z as:

$$zW_{opt} + (1-z)H_{opt} = \max_i \{zW_i + (1-z)H_i\}.$$

In the same year, 1951, Savage proposed a criterion containing the comparison of the result of every action with the results of other actions under the same state of nature [10].

Definition 7 (Savage criterion)**.** *Let the regret R_{ij} refers to the difference between the values coming from the best action given that S_j is the true state of nature and the value resulting from A_i under S_j:*

$$R_{ij} = \max_i \{v_{ij}\} - v_{ij}.$$

By the regret matrix of R_{ij} values, every action should be assigned the index of worst regret Y_i resulting from action A_i:

$$Y_i = \max_j \{R_{ij}\}$$

Savage's minimax regret criterion is defined as:

$$Y_{opt} = \min_i \{Y_i\}.$$

2. Extension of Decision Criteria to Three Dimensions

In this section, we present the extensions of the characteristic axioms of the criteria in three dimensions. Then, we give the definitions for the Laplace, Wald, Hurwicz, and Savage criteria in 3D and prove the related axioms of each criterion.

2.1. Axiomatic Characterization of Criteria in Three Dimensions

We extend and present the characteristic axioms presented by Milnor in [5], which characterize the decision criteria in decision theory, in 3D. These axioms are also mentioned in Pataki's paper [11]. First of all, we introduce the definitions of the row-block and column-block used throughout the extensions for 3D matrices.

Definition 8. *A row-block of a three-dimensional matrix A is the horizontal section of the matrix $A \in \mathbb{R}^{m \times n \times s}$ such that the r^{th} row-block is denoted by $A(r, j, k)$, where r is a fixed number in $[1, m]$ and $j = 1, ..., n$ and $k = 1, ..., s$. Similarly, a column-block of a three-dimensional matrix A is the vertical section of the 3D matrix A such that the t^{th} column-block is denoted by $A(i, t, k)$, where t is a fixed number in $[1, n]$, $i = 1, ..., m$, and $k = 1, ..., s$.*

Axiom 1 (ordering). *The relation is complete ordering. In other words, the following two laws are satisfied:*

- *For any two row-blocks r and r', either $r \geq r'$ or $r' \geq r$.*
- *If $r \geq r' \geq r''$ holds for r, r', and r'' row-blocks, then $r \geq r''$.*

Axiom 2 (symmetry). *The relation between any two row-blocks is not changed by any permutation of the row-blocks.*

Axiom 3 (strong domination). *If each element of p^{th} row-block is greater than q^{th}, then $r_p \geq r_q$.*

Axiom 4 (continuity). *If a sequence of three-dimensional matrices P_i converges to the matrix P and if $r_i \geq r_i'$ for all i, then the limit row-blocks r and r' satisfy $r \geq r'$.*

Axiom 5 (linearity). *The ordering relation is not changed if each element a_{ij}^k of the 3D matrix A is replaced by $\lambda a_{ij}^k + \mu$ and $\lambda > 0$.*

Axiom 6 (row-block adjunction). *The order of any two row-blocks is not changed by the adjunction of a new row-block.*

Axiom 7 (column-block linearity). *The order of the row-blocks is not changed if a constant is added to a column-block.*

Axiom 8 (column-block duplication). *The ordering is not changed if a new column-block, identical to some old column, is adjoined to the matrix.*

Axiom 9 (convexity). *If row-block r is equal to the average $\frac{1}{2}(r' + r'')$ of two equivalent row-blocks (i.e., $r' \sim r''$, if $r' \geq r''$ and $r'' \geq r'$), then $r \geq r'$.*

Axiom 10 (special row-block adjunction). *The ordering between the old row-blocks is not changed by the adjunction of a new row-block, providing that no component of this new row-block is greater than the corresponding components of all old row-blocks.*

In the following subsections, we present the extensions of the two-dimensional criteria to three dimensions.

2.2. Laplace Criterion in 3D

Definition 9. *Let $A \in \mathbb{R}^{m \times n \times s}$ be a three-dimensional matrix with entries v_{ij}^k. The Laplace criterion in 3D is defined as*

$$E_i^k = \sum_{j=1}^{n} p_{ij}^k v_{ij}^k$$

where p_{ij}^k denotes the probabilities of the each state of nature for $i = 1, 2, ..., m$ and $k = 1, ..., s$. Then, the Laplace criterion implies the action with

$$A_{opt} = \max_{i=1,...,m} \{E_i\}$$

where $E_i = \sum\limits_{k=1}^{s} E_i^k$ for all i=1, ..., m.

For simplicity, as in [5], if we assume the probabilities of the states of nature are equally likely, we represent the Laplace criterion for a matrix A in 3D, which is the extended form of the definition given by Laplace in [8], as a maximum of $r_i = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n} v_{ij}^k$ over i, where r_i represents the i^{th} row-block of 3D matrix A.

Proposition 1. *The Laplace criterion satisfies the ordering axiom.*

Proof. Let $A \in \mathbb{R}^{m \times n \times s}$ and $r_i = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n} v_{ij}^k$. Since $r_i \in \mathbb{R}$, it is obvious that the relation has complete ordering. Hence, the relation in the row-blockwise sense is a complete ordering. □

Proposition 2. *The Laplace criterion fulfills the symmetry axiom.*

Proof. Assume $r_i \geq r_j$ holds, where $1 \leq i, j \leq m$. Let us interchange the order of the row-blocks in the 3D matrix A as $i \leftrightarrow j$. Then, the relation turns out to be $r_{j'} \geq r_{i'}$, where $j' = i$ and $i' = j$ in the original matrix. The only change occurs in the notation in this case, but not in the relation sense. □

Proposition 3. *The Laplace criterion satisfies the strong domination axiom.*

Proof. Assume that $A(p,j,k) - A(q,j,k) > 0$ while $1 \leq p, q \leq m$ and for all $1 \leq j \leq n$, $1 \leq k \leq s$. Since $r_p = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n} A(p,j,k)$ and $r_q = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n} A(q,j,k)$ by the definition, then, $r_p - r_q = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}(A(p,j,k) - A(q,j,k)) > 0$. Thus, the result follows. □

Proposition 4. *The Laplace criterion satisfies the continuity axiom.*

Proof. Let P_i be a sequence of 3D matrices and $P \in \mathbb{R}^{m \times n \times s}$ a three-dimensional matrix while $\lim_{i\to\infty} P_i = P$. Furthermore, assume that $r_i^k \geq \bar{r}_i$, $\lim_{i\to\infty} r_i^k = r^k$, and $\lim_{i\to\infty} \bar{r}_i = \bar{r}$ hold for all k, where r^k and $\bar{r}$ are the row-blocks of P_i and P, respectively. Since the row-blocks are two-dimensional matrices and the Laplace criterion holds for the continuity axiom in 2D, the result follows: $r^k \geq \bar{r}$. □

Proposition 5. *The Laplace criterion holds for the linearity axiom.*

Proof. Assume that $r_p \geq r_q$ holds for any $1 \leq p, q \leq m$. Then, let us replace each a_{ij}^k of 3D matrix A with $\lambda a_{ij}^k + \mu$ where $\lambda > 0$. Accordingly,

$$\begin{aligned}
\bar{r}_p &= \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}\left[\lambda A(p,j,k) + \mu\right] \\
&= \left(\frac{\lambda}{ns}\left[\sum_{k=1}^{s}\sum_{j=1}^{n} A(p,j,k)\right] + \mu\right) \\
&\geq \left(\frac{\lambda}{ns}\left[\sum_{k=1}^{s}\sum_{j=1}^{n} A(q,j,k)\right] + \mu\right) = \bar{r}_q
\end{aligned}$$

holds by the assumption. Thus, $\bar{r}_p = \lambda r_p + \mu \geq \lambda r_q + \mu = \bar{r}_q$, and the proof is complete. □

Proposition 6. *The Laplace criterion holds for the row-block adjunction.*

Proof. Assume $r_i \geq r_j$. Let us add r_t to the matrix. Since there is a complete ordering, we can say $(r_i \geq r_t$ or $r_t \geq r_i)$ and $(r_j \geq r_t$ or $r_t \geq r_j)$. Then:
Case I. $r_i \geq r_t$ and $r_j \geq r_t \rightarrow r_i \geq r_j \geq r_t \geq \rightarrow r_i \geq r_j$.
Case II. $r_i \geq r_t$ and $r_t \geq r_j \rightarrow r_i \geq r_t \geq r_j \rightarrow r_i \geq r_j$.
Case III. $r_t \geq r_i$ and $r_j \geq r_t \rightarrow r_j \geq r_t \geq r_i$. Since we assume $r_i \geq r_j$, this case is not valid due to the violation of the assumption.
Case IV. $r_t \geq r_i$ and $r_t \geq r_j \rightarrow r_t \geq r_i \geq r_j$. By the assumption, the result is trivial. □

Proposition 7. *The Laplace criterion holds for the column-block linearity.*

Proof. Let us redefine the column-perturbed version of the 3D matrix A with $\bar{A}$ by adding any constant $c \in \mathbb{R}$ to each element in the t^{th} column of A (i.e., $\bar{A}(:,t,:) = A(:,t,:) + c$; here, the colon ":" refers to all the elements in the corresponding places). Now, assume that $r_p \geq r_q$

holds for 3D matrix A. Then, $r_p = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}\bar{A}(p,j,k) \geq \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}\bar{A}(q,j,k) = r_q$ also operates for $\bar{A}$, and the proof is complete. □

Proposition 8. *The column-block duplication axiom is not valid for the Laplace criterion.*

Proof. Since this axiom is not valid for the Laplace criterion in the two-dimensional case and by Definition 9, the axiom is not valid for the Laplace criterion in 3D as well. □

Proposition 9. *The Laplace criterion holds for the convexity.*

Proof. Assume that $r = \frac{1}{2}(r' + r'')$, where r' and r'' are equivalent row-blocks while $r = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}A(i,j,k)$, $r' = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}A(i^{'},j,k)$, and $r'' = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}A(i^{''},j,k)$ for fixed $i, i^{'}$ and $i^{''}$ in $[1,m]$. Then, we have

$$r = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}A(i,j,k) = \frac{1}{2ns}\left[\sum_{k=1}^{s}\sum_{j=1}^{n}A(i^{'},j,k) + A(i^{''},j,k)\right].$$

On the other hand,

$$\begin{aligned} r - r' &= \frac{1}{2ns}\left[\sum_{k=1}^{s}\sum_{j=1}^{n}A(i^{'},j,k) + A(i^{''},j,k)\right] - \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n}A(i^{'},j,k) \\ &= \frac{1}{2ns}\sum_{k=1}^{s}\sum_{j=1}^{n}\Big(A(i^{''},j,k) - A(i^{'},j,k)\Big) \\ &\geq 0 \qquad \text{(by the definition of the row equivalence)} \end{aligned}$$

and the proof is complete. □

Proposition 10. *The Laplace criterion holds for the special row-block adjunction axiom.*

Proof. We skip the proof since it is similar to the proof of Proposition 6. □

According to the definition of Laplace in [8,45], we present the new definition of the Laplace criterion in terms of the ∞-norm of the matrix in 3D as follows:

Definition 10 ([31,32])**.** *Let $A \in \mathbb{R}^{m\times n\times s}$ be a 3D matrix and the ∞-norm of A be defined as follows:*

$$||A||_\infty = \max_{1\leq i\leq m}\sum_{k=1}^{s}\sum_{j=1}^{n}|v_{ij}^{k}| = \textit{the largest absolute block-row sum.}$$

Definition 11. *Let $A \in \mathbb{R}^{m\times n\times s}$ be a 3D matrix with non-negative entries, then the Laplace criterion implies the action with $||A||_\infty$, where $||A||_\infty$ is the largest absolute row-block sum for the three-dimensional matrix.*

Remark 1. *It is important to notice that the Laplace criterion is invariant under the perturbation of the matrix by Proposition 5. In order to use the new definition of the Laplace criterion in terms of the infinity norm for a 3D matrix with some negative entries, we firstly get rid of the negative entries in the matrix by adding $||A||_{max}$, which is the largest absolute value of the element A, to each entry of the matrix.*

Remark 2. *Definition 11 is also valid for two-dimensional matrices since the linearity axiom holds in two dimensions, as well.*

2.3. Wald Criterion in 3D

Definition 12. *Let $A \in \mathbb{R}^{m\times n\times s}$ be a 3D matrix with entries v_{ij}^k. The Wald criterion is defined in three dimensions as*

$$\begin{aligned} W_i^k &= \min_j\{v_{ij}^k\} \\ W_i &= \min_k\{W_i^k\} \\ W_{opt} &= \max_i\{W_i\} \end{aligned}$$

where $i = 1, ..., m$, $j = 1, ..., n$, and $k = 1, ..., s$.

The characteristic axioms of the Wald criterion are presented in the following propositions.

Proposition 11. *The Wald criterion in three dimensions satisfies Axioms 1, 2, 4–6, and 10.*

Proof. Since the proofs are trivial, we skip them. □

Proposition 12. *The Wald criterion in three dimensions does not hold for the column-block linearity axiom.*

Proof. Since the order of the row may change by adding a constant to a column of the 2D matrix (i.e., the Wald criterion does not have the column linearity property in 2D) in terms of the Wald criterion, the axiom is not valid in 3D either, concerning the definition of the Wald criterion. (See Appendix A for an illustration). □

Proposition 13. *The Wald criterion in three dimensions satisfies Axioms 3 and 8.*

Proof of Axiom 3. If each element of the p^{th} row-block is greater than the q^{th} row-block's, then we have $r_p = \max(W_p) \geq \max(W_q) = r_q$. □

Proof of Axiom 8. Let r_p and r_q be any two row-blocks of the 3D matrix A while $r_p \geq r_q$. Since the addition of the duplicated column-block will not make any significant sense for the result in view of the 3D Wald criterion definition, the relation will be kept. □

Proposition 14. *The Wald criterion in 3D is suited to the convexity axiom.*

Proof. By considering the definition of the Wald criterion in three dimensions, let A be a 3D matrix in $\mathbb{R}^{m\times n\times s}$, and suppose that r' and r'' are equivalent row-blocks such that $r = \frac{1}{2}(r' + r'')$, while r, r' and r'' represent the i_0-th, $i^{'}$-th, and $i^{''}$-th row-blocks of matrix A, where $1 \leq i_0, i^{'}, i^{''} \leq m$. According to the assumptions and the definition of the Wald criterion in 3D, we have

$$\begin{aligned} r - r' &= \min_k \min_j (A(i_0, j, k)) - \min_k \min_j (A(i^{'}, j, k)) \\ &= \frac{1}{2} \min_k \min_j (A(i^{'}, j, k) + A(i^{''}, j, k)) - \min_k \min_j (A(i^{'}, j, k)) \\ &\geq \frac{1}{2}\left(\min_k \min_j (A(i^{'}, j, k)) + \min_k \min_j (A(i^{''}, j, k)) \right) - \min_k \min_j (A(i^{'}, j, k)) \\ &= \frac{1}{2} \min_k \min_j (A(i^{''}, j, k)) - \frac{1}{2} \min_k \min_j (A(i^{'}, j, k)) \\ &\geq 0 \text{ since } r' \sim r''. \end{aligned}$$

This concludes the proof. □

2.4. Hurwicz Criterion in 3D

Definition 13. *Let $A \in \mathbb{R}^{m \times n \times s}$ be a 3D matrix with entries v_{ij}^k. The Hurwicz criterion is defined in three dimensions as*

$$\begin{aligned} H_i^k &= \max_j \{v_{ij}^k\} \\ H_i &= \max_k \{H_i^k\} \\ H_{opt} &= \max_i \{H_i\} \end{aligned}$$

where $i = 1, ..., m$, $j = 1, ..., n$, and $k = 1, ..., s$.

According to the above definition, we recommend the Hurwicz criterion decision rule in 3D as:

$$zW_{opt} + (1-z)H_{opt} = \max_i \{zW_i + (1-z)H_i\}$$

which is based on Hurwicz's decision rule in 2D. Here, the constant $0 \leq z \leq 1$ measures the player's optimism. Note that the special case of the rule in 3D, while $z = 1$, is also identical to the Wald criterion in three dimensions.

Proposition 15. *The Hurwicz criterion in three dimensions holds Axioms 1, 2, 3, 4, 5, 6, 8, and 10.*

Proof. The proof steps of Axioms 1–6, 8, and 10 are almost the same as their respective proofs for the Laplace and/or Wald criteria. Therefore, they are skipped here. □

Proposition 16. *The Hurwicz criterion in three dimensions does not hold for Axioms 7 and 9.*

Proof. By using the fact that arises between 2D and 3D matrices, Axioms 7 and 9 do not hold in 3D since they do not promise anything in 2D either. (See Appendix A for the illustrations of the related cases). □

2.5. Savage Criterion in 3D

Definition 14. *Let $A \in \mathbb{R}^{m \times n \times s}$ be a 3D matrix with entries v_{ij}^k. The Savage criterion is defined in three dimensions as*

$$\begin{aligned} R_{ij}^k &= \max_k \max_i \{v_{ij}^k\} - v_{ij}^k \\ Y_i &= \max_k \max_j \{R_{ij}^k\} \\ Y_{opt} &= \min_i Y_i \end{aligned}$$

where R_{ij}^k represents three-dimensional positive regret matrix of A for $i = 1, ..., m$, $j = 1, ..., n$, and $k = 1, ..., s$.

Proposition 17. *The Savage criterion in three dimensions satisfies Axioms 1–5 and 7–10.*

Proof. The proofs are trivial since they can be made by following similar steps as used in the corresponding proofs for the Laplace, Wald, or Hurwicz criterion. □

Proposition 18. *The Savage criterion in three dimensions does not hold for Axiom 6.*

Proof. Let A be a 3D matrix in $\mathbb{R}^{m \times n \times s}$. There is no doubt that the Savage criterion suggests one of the row-blocks between the first to m^{th} row-blocks that is referred to as Y_{opt} is obtained by the definition. For instance, to present the effect of the row-block adjunction, suppose that we add a new row-block, all entries of which are $\max_{i,j,k} v_{ij}^k$ (i.e., the maximum elements of matrix A) to the bottom of the matrix A, and generate a new matrix $\bar{A} \in \mathbb{R}^{(m+1)\times n\times s}$. Therefore, the Savage criterion will suggest one of the row-block between first and $(m+1)^{\text{th}}$ row-blocks of matrix $\bar{A}$. However, it is important to notice that all entries of the last row-block of the regret matrix of $\bar{A}$ are zeros with respect to Definition 14. Additionally, it is crucial to realize that the regret matrix contains non-negative entries. By taking this fact into account, the Savage criterion, in other words $\bar{Y}_{opt}$, suggests the adjunct row-block (i.e., the $(m+1)^{\text{th}}$ row-block of $\bar{A}$), which is not one of the row-block of matrix A. Consequently, the row-block adjunction changes the result of the Savage criterion. Hence, the axiom fails. □

Table 2 summarizes the characteristic axioms of each criterion in three dimensions. The criteria characterized by the related axioms are marked as X and the compatibility of the axioms is shown with $\boxed{X}$ in the table. For example, the Laplace criterion in 3D is characterized by the following axioms: ordering, symmetry, strong domination, row-block adjunction, and column-block linearity. It does not satisfy the column-block duplication, and it has compatibility in terms of continuity, linearity, convexity, and special row-block adjunction.

Table 2. The characteristic axioms of 3-dimensional decision criteria.

Axioms	Laplace	Wald	Hurwicz	Savage
1. Ordering	X	X	X	X
2. Symmetry	X	X	X	X
3. Str. Domination	X	X	X	X
4. Continuity	$\boxed{X}$	X	X	X
5. Linearity	$\boxed{X}$	$\boxed{X}$	X	$\boxed{X}$
6. Row-Block Adjunction	X	X	X	
7. Col.-Block Linearity	X			X
8. Col.-Block Duplication		X	X	X
9. Convexity	$\boxed{X}$	X		X
10. Special Row-Block Adj.	$\boxed{X}$	$\boxed{X}$	$\boxed{X}$	X

Remark 3. *We would like to emphasize that all 3D criteria presented above can be reduced for the decision problems containing the 2D matrix only by choosing the third dimension parameter s as 1 in the related definitions.*

3. Examples: Multiple State Games

In this section, we present some useful illustrations of the criteria in three dimensions. The first example aims to show the usage of the criteria in detail. The second and third examples demonstrate the real-life applications of these criteria. The third dimension treats the time and speed of the wind in the second and third examples, respectively.

Example 1 (Fundamental example)**.** *The main purpose of this example is to show the application of the criteria in three dimensions comprehensively, as was done in [5,45] for two-dimensional matrices. Let A be a 3D matrix in $\mathbb{R}^{4\times4\times3}$, and define it as follows:*

$$A = \left[A^1 = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 7 & 7 & 5 & 6 \\ 6 & 6 & 6 & 6 \\ 5 & 9 & 5 & 5 \\ 6 & 8 & 5 & 5 \end{bmatrix}, A^3 = \begin{bmatrix} 4 & 4 & 2 & 3 \\ 3 & 3 & 3 & 3 \\ 2 & 6 & 2 & 2 \\ 3 & 5 & 2 & 2 \end{bmatrix} \right]$$

Figure 1 represents the 3D form of matrix A above. Each section in Figure 1 refers to the sections A^1, A^2, and A^3 from front to back, respectively. In other words, the entries of the most front section of Figure 1 are v^1_{ij} for $i = 1,2,3,4$ and $j = 1, 2, 3, 4$.

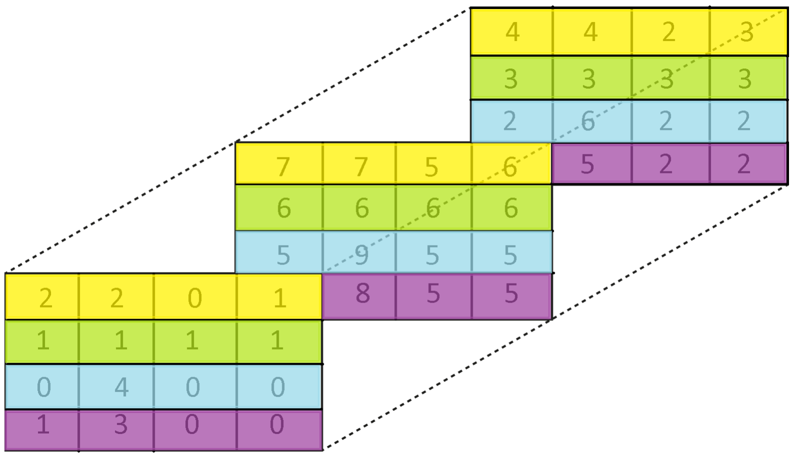

Figure 1. Extension form of the 3D matrix $A \in \mathbb{R}^{4\times4\times3}$ of Example 1.

Laplace criterion: We used the simplest form of the definition of the Laplace criterion in 3D as we may choose the row-block providing the highest average of the sum of the elements in the row-blocks. Therefore, we evaluated them by using $r_i = \frac{1}{ns}\sum_{k=1}^{s}\sum_{j=1}^{n} v_{ij}^k$ for $i = 1,2,3,4,$ so that

$$r_1 = \frac{1}{12}\sum_{k=1}^{3}\sum_{j=1}^{4} v_{ij}^k = 3.58,$$

and similarly $r_2 = 3.33$ and $r_3 = 3.33$, $r_4 = 3.33$ are calculated. Then,

$$A_{opt} = \max_i\{E_i\} = \max_i\{r_i\} = \{3.58, 3.33, 3.33, 3.33\} = 3.58$$

by the definition. Hence, the Laplace would suggest that the player choose the first row-block (i.e., Alternative 1).

On the other hand, we can easily evaluate the result of the Laplace criterion in 3D by using the 3D matrix norm definition presented in Definition 11 as

$$||A||_\infty = \max_{1\le i\le 4}\sum_{k=1}^{3}\sum_{j=1}^{4} |v_{ij}^k| = 43.$$

Similar to the original definition of the Laplace criterion in 3D, the norm definition also suggests Alternative 1. The advantage of using the 3D norm definition of the Laplace criterion is that the result is obtained quickly without any intermediate calculation steps.

Wald criterion: We use the definition of the Wald criterion and follow the steps of the definition in three dimensions as

$$\begin{aligned} W_i^k &= \min_j\{v_{ij}^k\} = \{(0,1,0,0)^T, (5,6,5,5)^T, (2,3,2,2)^T\} \\ W_i &= \min_k\{W_i^k\} = \{0,1,0,0\} \\ W_{opt} &= \max_i\{W_i\} = 1. \end{aligned}$$

Thus, the Wald criterion in 3D recommends preferring Alternative 2. In other words, the Wald criterion says that the best decision is the second row-block.

Hurwicz criterion: As we did in the analysis of the Wald criterion above, we follow the definition of the Hurwicz criterion in three dimensions and obtain the following:

$$\begin{aligned} H_i^k &= \max_j \{v_{ij}^k\} = \{(2,1,4,3)^T, (7,6,9,8)^T, (4,3,6,5)^T\} \\ H_i &= \max_k \{H_i^k\} = \{7,6,9,8\} \\ H_{opt} &= \max_i \{H_i\} = 9. \end{aligned}$$

In order to compare the results for different optimism index $z = \frac{3}{4}$, which is the indecision point for this example between the 2nd and 3rd row-blocks, we first consider the case while $z < \frac{3}{4}$; as an example, we chose $z = 0.5$, then we have

$$zW_{opt} + (1-z)H_{opt} = \max_i \{zW_i + (1-z)H_i\} = \max_i \{3.5, 3.5, 4.5, 4\} = 4.5.$$

According to the Hurwicz criterion, the best alternative is Alternative 3 (i.e., the third row-block) while $z = 0.5$. We now examine the case when $z > \frac{3}{4}$. As an illustration, let $z = 0.8$.

$$\begin{aligned} zW_{opt} + (1-z)H_{opt} &= \max_i \{zW_i + (1-z)H_i\} \\ &= \max_i \{1.4, 2, 1.8, 1.6\} = 2. \end{aligned}$$

Hence, the Hurwicz criterion advises preferring Alternative 2 (i.e., the second row-block) for $z = 0.8$.

Savage criterion: We firstly obtain the positive regret matrix R by using $R_{ij}^k = \max_k \max_i \{v_{ij}^k\} - v_{ij}^k$ in Definition 14 as

$$R = \left[R^1 = \begin{bmatrix} 5 & 7 & 6 & 5 \\ 6 & 8 & 5 & 5 \\ 7 & 5 & 6 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix}, R^2 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, R^3 = \begin{bmatrix} 3 & 5 & 4 & 3 \\ 4 & 6 & 3 & 3 \\ 5 & 3 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \right].$$

Then, we use the rest of the definition as

$$Y_i = \max_k \max_j \{R_{ij}^k\} = \{(7,8,7,6)^T\}$$

and

$$Y_{opt} = \min_i Y_i = 6.$$

Hence, the best decision is Alternative 4 according to the Savage criterion in 3D. In other words, Savage recommends the fourth row-block to the player.

We can summarize the obtained results via three-dimensional representation in Figure 2, which shows the composite form of the 3D matrix A. For example, the Laplace criterion suggests choosing Alternative 1. In other words, the yellow row-block refers to the result of the Laplace criterion in 3D. Similarly, the green, blue, and purple row-blocks refer to the results of the Wald, Hurwicz (for $z < \frac{3}{4}$), and Savage criteria, respectively.

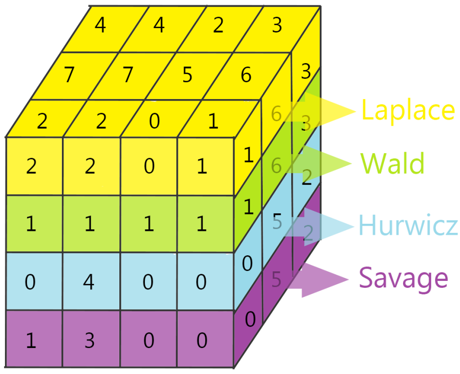

Figure 2. The compact form of the 3D matrix $A \in \mathbb{R}^{4\times4\times3}$ of Example 1 and the suggested solutions with 3D decision criteria.

Example 2 (COVID-19 stages with real data)**.** *Let $A \in \mathbb{R}^{2\times2\times3}$ be 3D matrix, which is created by using real data for South Korea [46], and A^k denotes the stages of the COVID-19 pandemic as the start, spread, and end for $k = 1, 2,$ and 3, respectively. In this modeling, the actions are keeping the quarantine and breaking the quarantine while the state of nature is the different infection risk of the individuals in the stages. Accordingly, we have the following 3D matrix A:*

$$A = \left[A^1 = \begin{bmatrix} 0.984 & 0.984 \\ -0.016 & -0.032 \end{bmatrix}, A^2 = \begin{bmatrix} 0.9634 & 0.9634 \\ -0.0366 & -0.0732 \end{bmatrix}, A^3 = \begin{bmatrix} 0.987 & 0.987 \\ -0.013 & -0.026 \end{bmatrix} \right]$$

Laplace criterion: We used the alternative definition of the Laplace criterion for the 3D matrix as follows:

$$\begin{aligned} r_1 &= \frac{1}{6} \sum_{k=1}^{3} \sum_{j=1}^{2} A(1, j, k) \\ &= \frac{1}{6}[(0.984 + 0.984) + (0.9634 + 0.9634) + (0.987 + 0.987)] \\ &= 0.9781. \end{aligned}$$

Similarly, we evaluated $r_2 = -0.0328$ as above. Then, $A_{opt} = \max_i\{E_i\} = \max_i\{r_i\} = 0.9781$. Therefore, the Laplace criterion suggests the first row-block, that is keeping the quarantine is the best alternative.
On the other hand, we can easily evaluate the Laplace criterion by Definition 11. However, we need to perturb the matrix by adding a 3D matrix with all entries of 0.987 in order to obtain a non-negative matrix in light of Remark 1. By the norm definition of the Laplace criterion applied on the perturbed 3D matrix, we have

$$||A||_\infty = \max_{1\le i\le 2} \sum_{k=1}^{3} \sum_{j=1}^{2} |v_{ij}^k| = 11.7908$$

that points out to the first row-block.

Wald criterion: In order to determine which row-block suggested by the Wald criterion, we applied Definition 12 step by step and obtain the following:

$$\begin{aligned} W_i^k &= \min_j\{v_{ij}^k\} = \{(0.984, -0.032)^T, (0.9634, -0.0732)^T, (0.987, -0.026)^T\} \\ W_i &= \min_k\{W_i^k\} = \{0.9634, -0.0732\} \\ W_{opt} &= \max_i\{W_i\} = 0.9634. \end{aligned}$$

Consequently, it is clear by the W_{opt} value that the Wald criterion implies Alternative 1/the first row-block is the best option.

Hurwicz criterion: Similarly, we used Definition 13 by choosing the optimism index $z = 0.5$ as an example, then

$$\begin{aligned} H_i^k &= \max_j \{v_{ij}^k\} = \{(0.984, -0.016)^T, (0.9634, -0.0366)^T, (0.987, -0.013)^T\} \\ H_i &= \max_k \{H_i^k\} = \{0.987, -0.013\} \\ H_{opt} &= \max_i \{H_i\} = 0.987. \end{aligned}$$

while

$$\begin{aligned} zW_{opt} + (1-z)H_{opt} &= \max_i \{zW_i + (1-z)H_i\} \\ &= \max_i \{0.9752, -0.0431\} = 0.9752. \end{aligned}$$

Hence, the Hurwicz criterion advises selecting Alternative 1. In other words, according to the Hurwicz criterion, the best option for the player is the first row-block.

Savage criterion: Firstly, we created the regret matrix using $R_{ij}^k = \max_k \max_i \{v_{ij}^k\} - v_{ij}^k$ and obtain

$$R = \left[R^1 = \begin{bmatrix} 0.003 & 0.003 \\ 1.003 & 1.019 \end{bmatrix}, R^2 = \begin{bmatrix} 0.0236 & 0.0236 \\ 1.0236 & 1.0602 \end{bmatrix}, R^3 = \begin{bmatrix} 0 & 0 \\ 1 & 1.013 \end{bmatrix} \right]$$

Next, we followed the rest of the definition of the Savage criterion in three dimensions as

$$\begin{aligned} Y_i &= \max_k \max_j \{R_{ij}^k\} \\ Y_{opt} &= \min_i Y_i = \min_i \{0.0236, 1.0602\} \\ Y_{opt} &= 0.0236. \end{aligned}$$

Thus, Savage suggests the best alternative as Alternative 1 (i.e., the first row-block).

It is well known in the literature that the combination of wind and hailstorms increases the risk of a lower harvest of fruits [47,48]. In light of this fact, we reconsidered the problem given in [12] and extended it to the three-dimensional case to examine the simultaneous effects of the hailstorm and wind speed on the harvest of apricot fruit in the following example under the perspective of decision criteria for the necessity of agricultural insurance.

Example 3 (Agricultural insurance with hailstorms and the speed of the wind)**.** *In this example, we used the data for the hailstorm given in [47] and for the effect of the wind in [48]. We considered that the loss of the farmer would be doubled when the wind speed doubled during the hailstorm since the hailstorm and the wind speed affect the production of apricot fruit in the field. We assumed that the maximum loss of the farmer would be −20,000 since the total value of the harvest is 20,000 in [47]. Then, the 3D payoff matrix $A \in \mathbb{R}^{2\times2\times3}$ is generated as follows:*

Firstcolumn-block and second column-block: "Hail storm does not exist" and "Hail storm exists"
First row-block and second row-block: "Insurance unavailable" and "Insurance available"

$$A = \left[A^1 = \begin{bmatrix} 0 & -5000 \\ -6400 & -6400 \end{bmatrix}, A^2 = \begin{bmatrix} -3200 & -10000 \\ -6400 & -6400 \end{bmatrix}, A^3 = \begin{bmatrix} -6400 & -20000 \\ -6400 & -6400 \end{bmatrix} \right]$$

Here, A^ks, for $k = 1, 2, 3$, represent the case for different wind speeds with the hailstorm situations. More explicitly, A^1, A^2, and A^3 refer to the situations with the wind speeds of 0, 20, and 40 kph, respectively, and the hailstorm cases.

Laplace criterion: We applied the definition of the Laplace criterion in 3D and evaluated $r_1 = -44600$ and $r_2 = -38400$. Then, $A_{opt} = \max_i\{E_i\} = \max_i\{r_i\} = -38400$. Therefore, the Laplace criterion suggests the best alternative to be the second row-block.

We can also recalculate the criterion result with Definition 11 by using the fact in Remark 1, which means we added the matrix A with a 3D matrix with all entries of 20000, as $||A||_\infty = 81600$, which says the best option is the insurance available action.

Wald criterion: By the Wald criterion stated in Definition 12, we have

$$\begin{aligned} W_i^k &= \min_j\{v_{ij}^k\} = \{(-5000, -6400)^T, (-10000, -6400)^T, (-20000, -6400)^T\} \\ W_i &= \min_k\{W_i^k\} = \{-20000, -6400\} \\ W_{opt} &= \max_i\{W_i\} = -6400. \end{aligned}$$

The best action is the second row-block according to the Wald criterion result.

Hurwicz criterion: We directly applied the steps of the Hurwicz criterion given in Definition 13 with the optimism index, for instance, $z = 0.6$, and we obtain

$$\begin{aligned} H_i^k &= \max_j\{v_{ij}^k\} = \{(0, -6400)^T, (-3200, -6400)^T, (-6400, -6400)^T\} \\ H_i &= \max_k\{H_i^k\} = \{0, -6400\} \\ H_{opt} &= \max_i\{H_i\} = 0. \end{aligned}$$

Furthermore, we have the following via the recommended Hurwicz criterion decision rule in 3D:

$$\begin{aligned} zW_{opt} + (1-z)H_{opt} &= \max_i\{zW_i + (1-z)H_i\} \\ &= \max_i\{-12000, -6400\} = -6400 \end{aligned}$$

Thus, the Hurwicz criterion encourages preferring the action of insurance available.

Savage criterion: By the Savage criterion in 3D, we first obtain the regret matrix R as

$$R = \left[R^1 = \begin{bmatrix} 0 & 0 \\ 6400 & 1400 \end{bmatrix}, R^2 = \begin{bmatrix} 3200 & 5000 \\ 6400 & 1400 \end{bmatrix}, R^3 = \begin{bmatrix} 6400 & 15000 \\ 6400 & 1400 \end{bmatrix} \right]$$

Then, we obtain from the rest of the steps the following:

$$\begin{aligned} Y_i &= \{15000, 6400\} \\ Y_{opt} &= 6400 \end{aligned}$$

The best alternative is the second row-block with respect to the Savage criterion.

4. Conclusions and Discussions

The main goal of these extensions is to discover the simultaneous effects of two states of nature on the decision process in 3D. For this purpose, we extended the Laplace, Wald, Hurwicz, and Savage criteria to three dimensions under uncertainty. In addition to these definitions, we presented a new definition for the Laplace criterion, based on only the ∞-norm of the 3D payoff matrix, which reduces the computational costs and obtains results quickly.

Then, we presented all 3D characteristic axioms for the Laplace, Wald, Hurwicz, and Savage criteria to demonstrate the effects of the third dimension theoretically. After giving the proofs of the propositions for all criteria, we stated that the criteria in three dimensions work well for any two-dimensional matrix if the third dimension is assumed as 1. Finally, we presented an illustrative fundamental example in order to show the application of the 3D decision criteria explicitly. Moreover, we showed the applicability of the 3D decision criteria on real-life problems, as well. We used the real data belonging to South Korea during the different stages of the first wave of the COVID-19 pandemic. Although it is hard to handle different stages' effects simultaneously in 3D matrix analyses, the 3D criteria helped us obtain the results easily, and they both suggested the best alternative during the COVID-19 pandemic to be the keeping the quarantine strategy while different factors were affecting the individuals' decision process.

Additionally, we investigated an agricultural insurance problem in 3D, which is based on natural events' (wind and hailstorm) effects, as another important real-life action. Under a similar complexity of the decision process in 3D, we simply concluded that it is necessary to obtain agricultural insurance for the apricot field in any weather condition with the guidance of the 3D criteria.

Consequently, we believe that the extensions of the criteria in 3D may have a wide application area. The usage of the criteria in three dimensions might especially be useful for some problems that contain two simultaneous states of nature. We also think that the applications of 3D matrices in decision and game theories may make it easier or take an important role to model some complex situations. For example, in the case of more than two states of nature, these cases cannot be modeled directly by the three-dimensional matrix. Therefore, these cases need higher-dimensional matrices, which require comprehensive work depending on the theoretical studies for the higher-dimensional matrices. Thus, the extensions and contributions of this paper could be the base for higher-dimensional extension research. On the other hand, since we focused on three-dimensional extensions of the criteria in this paper, this may be considered as a limitation of the study in view of theoretical and practical implications. The higher-dimensional extensions of the criteria are left for future research.

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Appendix A

Let us consider the following 3D matrix $A \in \mathbb{R}^{3\times3\times2}$ to illustrate the case of the unsatisfied characteristic axioms for the related criteria.

$$A = \left[A^1 = \begin{bmatrix} 0 & 4 & -1 \\ 2 & 3 & -1 \\ 4 & 2 & -1 \end{bmatrix}, A^2 = \begin{bmatrix} -3 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & -3 & -2 \end{bmatrix} \right].$$

Demonstration A1. The Wald criterion in 3D does not satisfy the column-block linearity axiom: The Wald criterion result can be obtained as $W_{opt} = -1$. Thus, the Wald criterion suggests preferring Alternative 2 (see the left-hand side of Figure A1).

To analyze the effect of the column linearity on the result, let us add a constant, for instance $c = -6$, to the second column-block of the matrix A. Then, we obtain

$$\bar{A} = \left[A^1 = \begin{bmatrix} 0 & -2 & -1 \\ 2 & -3 & -1 \\ 4 & -4 & -1 \end{bmatrix}, A^2 = \begin{bmatrix} -3 & -5 & 0 \\ -1 & -7 & -1 \\ 1 & -9 & -2 \end{bmatrix} \right].$$

Similarly, the criterion can be obtained as $\bar{W}_{opt} = -5$ for the matrix $\bar{A}$. Hence, the Wald criterion points out that the best action is the first row-block (see the right-hand side of Figure A1). Thus, it is clear that Axiom 7 fails for the Wald criterion.

Demonstration A2. The Hurwicz criterion fails to satisfy the column-block linearity axiom: After we evaluate W_i and H_i for the Wald and Hurwicz criteria, respectively, we have

$$zW_{opt} + (1-z)H_{opt} = \max_i \{zW_i + (1-z)H_i\} = \max_i \{-0.2, 0.6, -0.2\} = 0.6,$$

which implies the second-row block while $z = 0.6$ (see the left block of Figure A1). By following the same steps, we obtain $\bar{W}_i$ and $\bar{H}_i$ for the perturbed 3D matrix $\bar{A}$ above and

$$z\bar{W}_{opt} + (1-z)\bar{H}_{opt} = \max_i \{z\bar{W}_i + (1-z)\bar{H}_i\} = \max_i \{-3, -3.4, -3.8\} = -3,$$

which points to the first-row block for $z = 0.6$ (see the right block of Figure A1). Therefore, the Hurwicz criterion does not satisfy the axiom.

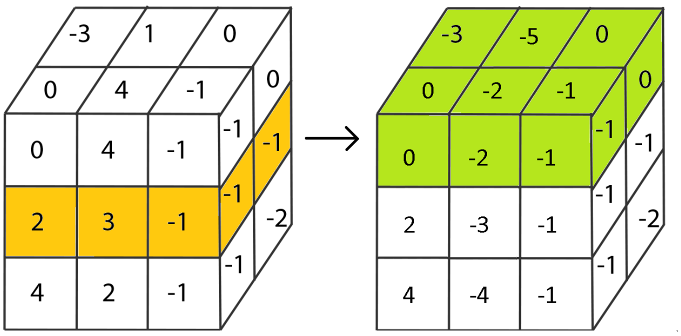

Figure A1. Column-block linearity analyses for Wald and Hurwicz criteria: change in the results via the 3D matrices A (on the **left**) and $\bar{A}$ (on the **right**).

Demonstration A3. The Hurwicz criterion does not satisfy the convexity axiom: In this demonstration, we again used the 3D matrix A above. The second row-block is equal to the average of the first and third row-blocks, which are equivalent row-blocks, of A. By the Hurwicz criterion, it is clear that $r_1 = r_3 = 4$ while $r_2 = 3$. Hence, the convexity axiom fails for the Hurwicz criterion.

References

1. Triantaphyllou, E. Multi-Criteria Decision Making Methods. In *Multi-Criteria Decision Making Methods: A Comparative Study*; Applied Optimization; Springer: Boston, MA, USA, 2000; Volume 44.
2. Bennet, A.; Bennet, D. The decision-making process in a complex situation. In *Handbook on Decision Support Systems 1*; Springer: Berlin/Heidelberg, Germany, 2008; pp. 3–20.
3. Vasilescu, C. Effective strategic decision-making. *J. Def. Resour. Manag.* **2011**, *2*, 101.
4. Taghavifard, M.T.; Damghani, K.K.; Moghaddam, R.T. Decision making under uncertain and risky situations. In *Enterprise Risk Management Symposium Monograph Society of Actuaries*; SOA: Schaumburg, IL, USA, 2009.
5. Milnor, J.W. *Game Against Nature*; RAND Corporation: Santa Monica, CA, USA, 1951.
6. Biswas, T. Games against Nature and the Role of Information in Decision-Making under Uncertainty. In *Decision-Making under Uncertainty*; Palgrave: London, UK, 1997.
7. Pazek, K.; Crtomir, R. Decision making under conditions of under certainty in agriculture: A case study of oil crops. *Osijek* **2009**, *15*, 154–176.
8. Laplace, P.S. *Theorie Analitique des Probabilites*; Courcier: Paris, France, 1812; Volume 7.
9. Hurwicz, L. Generalized bayes minimax principle: A criterion for decision-making under uncertainty. *Cowles Comm. Discuss. Pap. Stat.* **1951**, *355*, 1–7.
10. Savage, L.J. The theory of statistical decision. *J. Am. Stat. Assoc.* **1951**, *46*, 55–67. [CrossRef]
11. Pataki, B. A critique of some classical theories of decision under uncertainty. *Period. Polytech. Soc. Manag.* **1996**, *4*, 79–92.
12. İzgi, B.; Özkaya, M. The demonstration of the necessity of agriculture insurance by game theory: Matrix norm approach. *AKU J. Sci. Eng.* **2020**, *20*, 824–831. [CrossRef]
13. Kassa, A.M. Application of decision-making with uncertainty techniques: A case of production volume of maize in Ethiopia. *Int. J. Qual. Res.* **2017**, *11*, 331.
14. Ballestero, E.; Günther, M.; Pla-Santamaria, D.; Stummer, C. Portfolio selection under strict uncertainty: A multicriteria methodology and its application to the Frankfurt and Vienna stock exchange. *Eur. J. Oper. Res.* **2007**, *181*, 1479–1487. [CrossRef]
15. Ünal, C.; Atılgan, T. Decision making techniques in apparel industry. *Tekst. Konfeksiyon* **2007**, *17*, 256–263. (In Turkish)
16. Galasso, F.; Thierry, C. Design of cooperative processes in a customer–supplier relationship: An approach based on simulation and decision theory. *Eng. Appl. Artif. Intell.* **2009**, *22*, 865–881. [CrossRef]
17. Green, M.; Weatherhead, E.K. Coping with climate change uncertainty for adaptation planning: An improved criterion for decision-making under uncertainty using UKCP09. *Clim. Risk Manag.* **2014**, *1*, 63–75. [CrossRef]
18. Ma, J.; Harsvedt, J.D.; Jaradat, R.; Smith, B. Sustainability driven multicriteria project portfolio selection under uncertain decision-making environment. *Comput. Ind. Eng.* **2019**, *140*, 106236. [CrossRef]
19. Yazdani, M.; Tavana, M.; Pamucar, D.; Chatterjee, P. A rough based multicriteria evaluation method for healthcare waste disposal location decisions. *Comput. Ind. Eng.* **2020**, *143*, 106394. [CrossRef]
20. Saldanha, W.H.; Arrieta, F.R.P.; Ekel, P.I.; Machado-Coelho, T.M.; Soares, G.L. Multi-criteria decision-making under uncertainty conditions of a shell-and-tube heat exchanger. *Int. J. Heat Mass Transf.* **2020**, *155*, 119716. [CrossRef]
21. Ulansky, V.; Raza, A. Generalization of minimax and maximin criteria in a game against nature for the case of a partial a priori uncertainty. *Heliyon* **2021**, *7*, e07498.
22. Shmelova, T.; Rizun, N.; Kredentsar, S.; Yastrub, M. Enhancement of the Methodology for the Selection of the Optimal Location of the Remote Tower Centre. In *CITRisk*. 2021; pp. 335–346. Available online: https://ceur-ws.org/Vol-3101/Paper25.pdf (accessed on 9 October 2022).
23. Pak, M.; Botasheva, F.; Rachek, S. *Assessment of Transport Corridors Efficiency in the Arctic Zone*; Lecture Notes in Networks and Systems; Springer: Cham, Switzerland, 2022; Volume 403.
24. Shmelova, T.; Sechko, O. Hybrid Expert System for Collaborative Decision-Making in Transportation Services of Healthcare Needs. In *IDDM*. 2021; pp. 127–135. Available online: https://ceur-ws.org/Vol-3038/short6.pdf (accessed on 9 October 2022).
25. Wang, Y.; Liu, L.; Geng, Y.; Wang, C.; Li, S. Formation of a costs model for patient rehabilitation. *Iran. J. War Public Health* **2022**, *14*, 1–9.
26. Balasa, R.; Andrei, A.G.; Costea, M.L.; Apostol, E.I.; Semenscu, A. A dynamic approach to wind tunnel testing under risk conditions assessment. *J. Phys. Conf. Ser.* **2022**, *2212*, 012007. [CrossRef]
27. Gomes, M.I.; Martins, N.C. *Mathematical Models for Decision Making with Multiple Perspectives: An Introduction*; CRC Press: Boca Raton, FL, USA, 2022.
28. Balakina, G.; Kibalov, E.; Pyataev, M. System assessment of the Kyzyl-Kuragino railroad project as an element of Russia-Mongolia-China transport corridors. *Transp. Res. Procedia* **2022**, *63*, 1817–1825. [CrossRef]
29. Vdovyn, M.; Zomchak, L.; Panchyshyn, T. Modeling of Economic systems using game theory. *Věda Perspekt.* **2022**, *7*, 14.
30. Walker, L.; Hischier, I.; Schlueter, A. Scenario-based robustness assessment of building system life cycle performance. *Appl. Energy* **2022**, *311*, 118606. [CrossRef]
31. İzgi, B. Behavioral Classification of Stochastic Differential Equations in Mathematical Finance. Ph.D. Thesis, Istanbul Technical University, Istanbul, Turkey, 2015; 92p.
32. Duran, A.; İzgi, B. Application of the Heston stochastic volatility model for Borsa Istanbul using impression matrix norm. *J. Comput. Appl. Math.* **2015**, *281*, 126–134. [CrossRef]

33. İzgi, B.; Özkaya, M. Norm Inequalities and Applications in 3-Dimensional Matrices. *AKU J. Sci. Eng.* **2017**, *17*, 906–913. [CrossRef]
34. İzgi, B.; Özkaya, M. Fundamental concept of 3-D matrices. *Aip Conf. Proc.* **1991**, *2017*, 020008.
35. Csima, J. On the plane term rank of three dimensional matrices. *Discret. Math.* **2021**, *28*, 147–152. [CrossRef]
36. Burkard, R.E.; Rudolf, R.; Woeginger G.J. Three-dimensional axial assignment problems with decomposable cost coefficients. *Discret. Appl. Math.* **1996**, *65*, 123–139. [CrossRef]
37. Ignatova V.; Styczynsk Z. Statistical Matrix Representation of Time-Varying Electrical Signals. Reconsruction and Prediction Application. In Proceedings of the 9th International Conference on Probabilitstic Methods Applied to Power Systems KTH, Stockholm, Sweden, 11–15 June 2006,
38. Tao, L.; Chen, Y.; Liu, X.; Wang, X. An integrated multiple criteria decision-making model applying axiomatic fuzzy set theory. *Appl. Math. Model.* **2013**, *36*, 5046–5058. [CrossRef]
39. Lotfi, F.H.; Rostamy-Malkhalifeh, M.; Aghayi, N.; Beigi, Z.G.; Gholami, K. An improved method for ranking alternatives in multiple criteria decision analysis. *Appl. Math. Model.* **2013**, *37*, 25–33. [CrossRef]
40. Ju, Y. A new method for multiple criteria group decision-making with incomplete weight information under linguistic environment. *Appl. Math. Model.* **2014**, *38*, 5256–5268. [CrossRef]
41. Ishizaka, A.; Nemery, P. *Multi-Criteria Decision Analysis: Methods and Software*; Wiley: New Delhi, India, 2013.
42. Kauffmann, A. *The Science of Decision Making*; World Univeristy Library: London, UK, 1968.
43. Szentpeteri, S. *Gazdasagi döNtesek Bizontalansag Oseten*; Közgazdasagi es Jogi Könyviado: Budapest, Hungary, 1980.
44. Collingridge, D. *Critical Decision Making*; Frances Pinter: London, UK, 1982.
45. Straffin, P. *Game Theory and Strategy*; Mathematical Association of America: Washington, DC, USA, 1993.
46. Özkaya, M.; İzgi, B. Effects of the quarantine on the individuals' risk of COVID-19 infection: Game theoretical approach. *Alex. Eng. J.* **2021**, *60*, 4157–4165. [CrossRef]
47. HDI Sigorta. Devlet DESTEKLI Bitkisel ürün Sigortası. 2021. Available online: https://hdisigorta.com.tr (accessed on 13 August 2021). (In Turkish)
48. Republic of Türkiye Ministry of Agriculture and Forestry. Ürün Masalları-Kayısı Bülteni. 2021. Available online: https://www.tarimorman.gov.tr (accessed on 17 August 2021). (In Turkish)