

## Coherence resonance in a spatial prisoner's dilemma game

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**Abstract.** We study effects of additive spatiotemporal random variations, introduced to the payoffs of a spatial prisoner's dilemma game, on the evolution of cooperation. In the absence of explicit payoff variations the system exhibits a phase transition from a mixed state of cooperators and defectors to a homogenous state of defectors belonging to the directed percolation universality class. By introducing nonzero random variations to the payoffs, this phase transition can be reverted in a resonance-like manner depending on the variance of noise, thus marking coherence resonance in the system. We argue that explicit random payoff variations present a viable mechanism that promotes cooperation for defection temptation values substantially exceeding the one marking the transition point to homogeneity by deterministic payoffs.

### Contents

<b>1. Introduction</b>	<b>2</b>
<b>2. Spatial prisoner's dilemma game</b>	<b>3</b>
<b>3. Coherence resonance</b>	<b>4</b>
<b>4. Summary and discussion</b>	<b>6</b>
<b>References</b>	<b>7</b>

## 1. Introduction

The prisoner's dilemma game and its variations [1]–[4] present the workhorse for studying the evolution of altruistic behaviour among selfish individuals in human and animal societies. In its original form [5], the prisoner's dilemma game consists of two players who have to decide simultaneously whether they want to cooperate or defect. The dilemma is given by the fact that although mutual cooperation yields the highest collective payoff, which is equally shared among the two players, individual defectors will do better if the opponent decides to cooperate. Since selfish players are aware of this fact they both decide to defect, whereby none of them gets a profit. Thus, instead of equally sharing the rewarding collective payoff received by mutual cooperation, they end up empty-handed. This unfavourable result is in classical game theory known as the Nash equilibrium [6]. Although standing firm on mathematical proofs, the Nash equilibrium is in real life more of an exception rather than a fixed principle. Accordingly, several mechanisms have been proposed to explain the emergence of cooperation in various types of games. Among the more prominent are spatial extensions [7]–[16], direct and indirect reciprocity [2], [17]–[20] and voluntary participation [21]–[24].

Besides mechanisms that facilitate cooperation, stochastic processes are also acknowledged to play a vital part in evolutionary dynamics, affecting both the overall population gain [25] and equilibrium selection [26]–[28], or even the nature of phase transition from one equilibrium towards the other [9, 13]. In particular, Traulsen *et al* [25] have shown that additive noise introduced to the classical replicator dynamics [4], supplemented by adaptive learning rates, can enhance the average payoff of the population in a coherence-resonance-like manner. Moreover, noise introduced in the strategy adoption process of spatially distributed players [13] can induce phase transitions that fall under the directed percolation universality class [29]–[31], as well as maintain cooperation at the highest possible level with respect to other game parameters, as is for example the temptation to defect [28]. While the study of Traulsen *et al* [25] uses the replicator equations and thus assumes infinite populations, authors in [9, 13, 28] utilize finite size two-dimensional lattices of various connectivity structures.

The aim of the present study is to report a new noise-induced phenomenon in finite size populations on square two-dimensional lattices with a fixed number of neighbours. In particular, we study effects of temporally and spatially white additive Gaussian noise introduced in the payoff matrix of the spatially extended prisoner's dilemma game. We show that appropriate levels of noise can revert the extinction of cooperators in a resonant manner. Since no additional deterministic inputs are introduced to the system the reported phenomenon is thus conceptually identical to coherence resonance reported previously e.g. for excitable dynamical systems in time [32] and space [33], provided that the facilitation and maintenance of cooperation is considered a constructive effect. We explain the phenomenon via analogies with dynamical systems close to bifurcation points, where it is well known that noise can anticipate the behaviour of the system beyond the bifurcation. Moreover, since uncertainties are a part of everyday life, we argue that explicit random payoff variations present a viable mechanism that promotes cooperation in various environments, ranging from social and animal societies to economic cycles.

The paper is structured as follows. Section 2 is devoted to the description of the evolutionary spatial prisoner's dilemma game. In section 3 evidence for the coherence resonance is presented, while in the last section we summarize the results and outline possible real-life implications of our findings.

## 2. Spatial prisoner's dilemma game

For the following analyses, we consider an evolutionary prisoner's dilemma game with players located on vertices of a regular two-dimensional square lattice of size  $n \times n$  with periodic boundary conditions. Moreover, we assume that each individual interacts, i.e. plays the game, only with its four nearest neighbours located to the north, south, east and west, whereby self-interactions are excluded. Each player can decide either to cooperate ( $C$ ) or to defect ( $D$ ). Depending on the choice of their strategies, each two players ( $P_i, P_j$ ) can, at every interaction, receive payoffs summarized succinctly by the so-called payoff matrix [8]

$P_i/P_j$	$C$	$D$
$C$	$1 + \xi_i/1 + \xi_j$	$1 + r + \xi_i/-r + \xi_j$
$D$	$-r + \xi_i/1 + r + \xi_j$	$0 + \xi_i/0 + \xi_j$

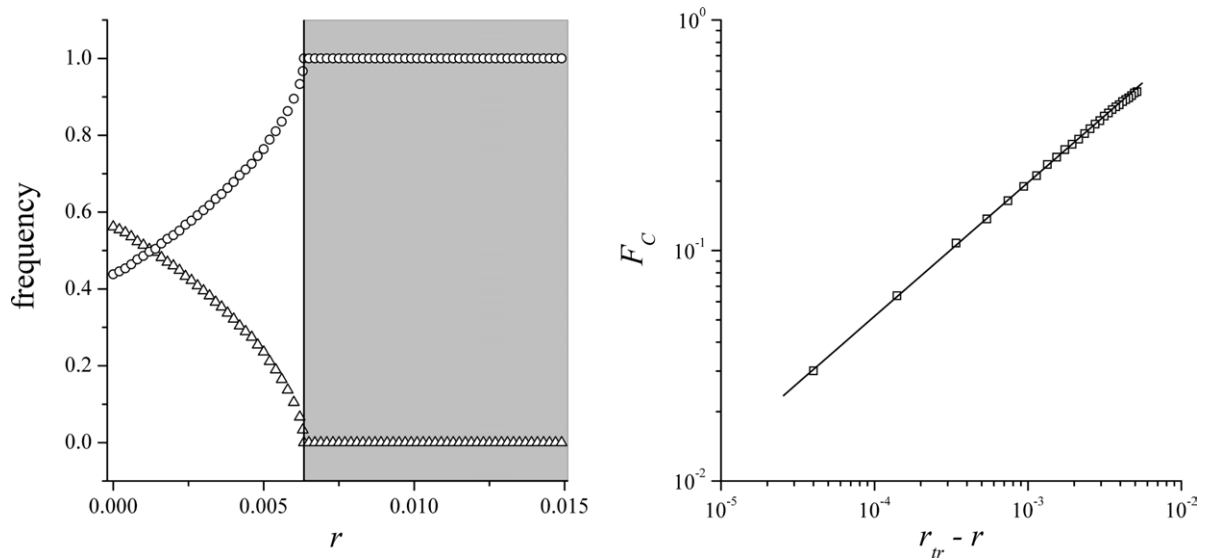
(1)

that is currently subjected to temporally and spatially white additive Gaussian noise, satisfying the correlation function  $\langle \xi_i(k)\xi_j(l) \rangle = \sigma^2 \delta_{ij} \delta_{kl}$ , whereby indices  $(i, j)$  mark any of the two neighbouring players, whilst  $k$  and  $l$  index two consecutive pair interactions. Moreover,  $r \geq 0$  determines the temptation to defect and  $\sigma^2$  is the variance of payoff variations. Starting from uniformly distributed cooperators and defectors on the square lattice, each player can adopt its strategy according to the performance of neighbouring players, whereby the probability that a player  $P_i$  will adopt the strategy of one of its randomly chosen nearest neighbours  $P_j$  is determined by the cumulative payoffs  $S_i$  and  $S_j$  of both players according to

$$W[P_i \leftarrow P_j] = \frac{1}{1 + \exp[(S_i - S_j)/K]}, \quad (2)$$

where  $K$  is the uncertainty related to the strategy adoption.  $0 < K \ll 1$  implies that the better performing player is readily adopted, whilst it is not completely impossible to adopt the strategy of a worse performing player. Importantly, although nonzero values of  $K$  can itself be considered as noisy disturbances acting on the system [28], we presently apply  $K = 0.1$  and the strategy adoption rule given by equation (2) solely to get smooth phase transitions at critical points [13], whilst otherwise qualitatively identical results as will be reported below can also be obtained by deterministic adoption rules where strategies of better and worse performing players are always and never adopted, respectively.

The described spatial prisoner's dilemma game can be iterated forward in time using either a synchronous or a random Monte Carlo update scheme [16], whereby it has been discovered that by non-deterministic evolutionary rules obtained results differ only slightly [34]. Since the synchronous update scheme tends to converge more quickly to the equilibrium than the random iteration, we thus apply the latter, letting all individuals interact pairwise with all their nearest neighbours and then simultaneously update their strategies according to equation (2). After every such iteration cycle, we set  $S_i = S_j = 0$  for  $\forall i, j$  and repeat the game. For a large enough number of iteration cycles ( $\geq 10^5$ ) and large system sizes ( $n \geq 400$ ), the frequencies of cooperators  $F_C$  and defectors  $F_D$ , satisfying  $F_C + F_D = 1$  at all times, approach an equilibrium value irrespective of the initial conditions. Presently,  $F_C$  and  $F_D$  are calculated by letting the system evolve for a long transient time towards the equilibrium, and then evaluate the average fraction of cooperators and defectors over an equally long time span following the discard time.

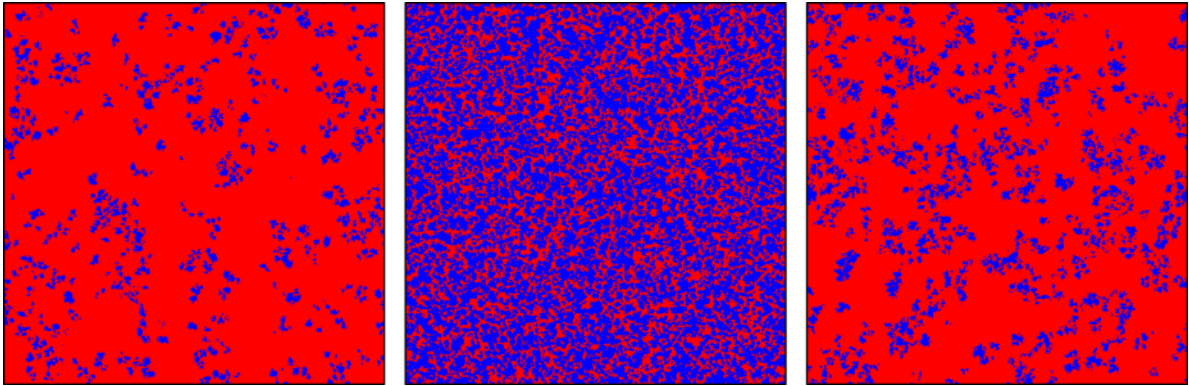


**Figure 1.** Left panel: frequencies of cooperators ( $\Delta$ ) and defectors ( $\circ$ ) for various  $r$  obtained by deterministic payoffs. The vertical line marks the transition point to homogeneity at  $r_{tr} = 0.00634(4)$ , whereby the greyed area marks the region of interest for  $\sigma > 0$  in subsequent calculations. Right panel:  $F_C$  in dependence on the distance to the phase transition. Note that both axes have a logarithmic scale, whereby a linear line with a slope  $\beta = 0.58(2)$  accurately fits to the numerically obtained data. Results were calculated for  $400 \leq n \leq 1000$  and  $10^5$ – $10^6$  iteration steps, depending on the proximity to the transition point  $r = r_{tr}$ .

Results presented in the left panel of figure 1 show the equilibrium frequencies  $F_C$  and  $F_D$  for various values of  $r$  obtained by deterministic payoffs ( $\sigma = 0$ ). It is evident that by small-enough values of  $r$  cooperators are able to survive since the risk of cooperation is low in comparison to the possible punishment. However, as the temptation to defect crosses a certain threshold  $r = r_{tr}$  cooperators die out ( $F_C = 0$ ), whereby the transition from the mixed to the homogenous state pertains to the directed percolation universality class [29]–[31] since  $F_C \propto (r_{tr} - r)^\beta$  by  $\beta = 0.58(2)$ , as shown in the right panel of figure 1. These are well-known results reported recently in [13, 22, 28], for example. In what follows, we will systematically analyse effects of nonzero  $\sigma$  on  $F_C$  and  $F_D$  for  $r > r_{tr}$ , with the aim of reporting noise-induced transitions to the mixed state in a resonance-like manner depending on  $\sigma$ , thus evidencing coherence resonance in the studied system.

### 3. Coherence resonance

We start the study by visually inspecting three characteristic spatial distributions of cooperators and defectors obtained by various values of  $\sigma$  for  $r$  slightly above  $r_{tr}$ . As evidenced in figure 1, zero values of  $\sigma$  for  $r > r_{tr}$  yield exclusive dominance of defectors given by  $F_D = 1$ . On the other hand, results presented in the panels of figure 2 clearly show that cooperation can be revitalized by nonzero values of  $\sigma$  in a resonant manner. In particular, while small  $\sigma$  are able to sustain only small clusters of cooperators scattered across the spatial grid, intermediate  $\sigma$

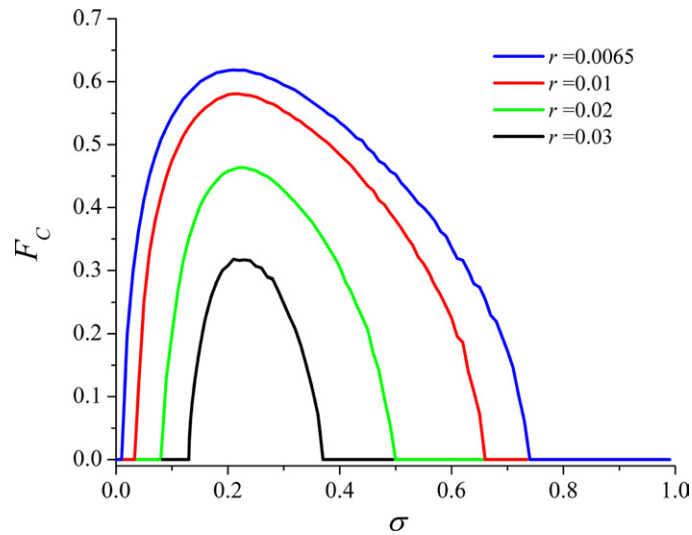


**Figure 2.** Characteristic equilibrium spatial distributions of cooperators (blue) and defectors (red) obtained by  $\sigma = 0.01$  ( $F_C = 0.091$ , left panel),  $\sigma = 0.20$  ( $F_C = 0.619$ , middle panel) and  $\sigma = 0.72$  ( $F_C = 0.148$ , right panel) for the defection temptation value  $r = 0.0065$ . All panels are depicted on a  $400 \times 400$  spatial grid.

can facilitate cooperation to the point of domination, or at least equality, implied by  $F_C \geq F_D$ . For larger values of  $\sigma$  the spatial distribution of the two strategies is again similar as by small  $\sigma$ . By considering the noise-induced maintenance and facilitation of cooperation a constructive effect, which is reasonable since widespread cooperators yield higher total population payoffs in comparison to dominant defectors and are therefore favourable for the society, results presented in figure 2 thus indicate a typical coherence resonance scenario [32, 33].

To quantify the ability of each particular  $\sigma$  to facilitate and maintain cooperation more precisely, we calculate  $F_C$  in dependence on various  $\sigma$  and  $r > r_{tr}$ . Moreover, we assume that  $F_C$  uniquely determines the constructive effects of noise on the system and thus has the same meaning as the signal-to-noise ratio [35] in classical stochastic and coherence resonance phenomena observed in dynamical systems. Results presented in figure 3 clearly evidence that there always exists an optimal level of additive spatiotemporal noise for which  $F_C$  is maximal, thus indicating the existence of coherence resonance in the studied spatial prisoner's dilemma game. Importantly, cooperation can be revitalized and maintained for defection temptation values substantially exceeding  $r_{tr}$ .

In order to shed light on the observed phenomenon, we draw analogies with dynamical systems, where it is long known that noise can anticipate the behaviour of the system waiting close to a bifurcation point in a resonant manner (see e.g. [36]). By devising a mean-field-like or pair approximation of the spatial prisoner's dilemma game [13, 15, 37], we easily end up with a set of ordinary differential equations, exhibiting rich dynamical behaviour depending on the approximation rules and the game under consideration [22, 23, 38]. We argue that the phase transition occurring at  $r = r_{tr}$  is conceptually identical to a bifurcation point of a dynamical system. Thus, noise can anticipate the behaviour on the other side of the bifurcation point in a resonant manner. This conjecture can be additionally strengthened by considering the fact that the parameter  $r$  in the noisy regime truly acts as a bifurcation parameter since increasing values, pushing the system further away from the transition (bifurcation) point  $r_{tr}$ , make it increasingly difficult for noise to anticipate the behaviour of the system beyond the bifurcation, as indicated by the decreasing maximal values of  $F_C$  for larger  $r$  in figure 3. However, since the mean-field-like



**Figure 3.** Coherence resonance in the studied spatial prisoner's dilemma game for various values of  $r$ .

or pair approximations of the spatial prisoner's dilemma game can be devised in numerous ways depending on the desired accuracy with the original model and its numerical implementation, we leave extensive explicit calculations for future studies. Finally, we note that by introducing also other strategies to the prisoner's dilemma game, such as for example the loners [21], bifurcation points also from steady state towards oscillatory behaviour can be obtained [38]. We therefore argue that the evolutionary dynamics of spatial games bears a treasure of possible noise-induced phenomena yet to be discovered.

#### 4. Summary and discussion

In summary, we show that spatially and temporally white additive Gaussian noise introduced in the payoff matrix of an evolutionary spatial prisoner's dilemma game can facilitate and maintain cooperation in a resonant manner depending on the level of random variations. This phenomenon can be observed for defection temptation values substantially exceeding the one marking the transition point to homogeneity by deterministic payoffs. By interpreting cooperation as a constructive strategy, the reported phenomenon is thus conceptually identical as coherence resonance reported previously in temporal and spatially extended dynamical systems [32, 33], [39]–[47], where exclusively random perturbation have been found to constructively affect the dynamics of the system under study.

We argue that random payoff variations are common in real life, either in human and animal societies or economic cycles. In particular, it is not difficult to imagine that a successful spreading of a community with certain beliefs or economic interests, for example, or a reproductive success of a species is affected by numerous unpredictable factors. It seems reasonable that the interaction phase between two individuals, each trying to make the best out of the encounter, is the most likely part of the evolutionary process for uncertainties to take effect. Thereby, unpredictable disturbances can arise either from the players themselves, for example by not adhering to the rules of the game in trying to make an illegal profit, or from the environment which can either



favour or hinder the success of each player. Although the presently applied approach assumes uncorrelated Gaussian distributed disturbances, future studies will be necessary to clarify the importance of temporal and spatial correlations as well as different distributions of such random influences.

## References

- [1] Maynard Smith J 1982 *Evolution and the Theory of Games* (Cambridge: Cambridge University Press)
- [2] Axelrod R 1984 *The Evolution of Cooperation* (New York: Basic Books)
- [3] Weibull J W 1995 *Evolutionary Game Theory* (Cambridge, MA: MIT Press)
- [4] Hofbauer J and Sigmund K 1998 *Evolutionary Games and Population Dynamics* (Cambridge: Cambridge University Press)
- [5] Neumann von J and Morgenstern O 1944 *Theory of Games and Economic Behavior* (Princeton, NJ: Princeton University Press)
- [6] Nash J 1950 *Econometrica* **18** 155
- [7] Nowak M A and May R M 1992 *Nature* **359** 826
- [8] Nowak M A and May R M 1993 *Int. J. Bifurcat. Chaos Appl. Sci. Eng.* **3** 35
- [9] Nowak M A, Bonhoeffer S and May R M 1994 *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **4** 33
- [10] Herz A V M 1994 *J. Theor. Biol.* **169** 65
- [11] Lindgren K and Nordahl M G 1994 *Physica D* **75** 292
- [12] Nakamaru M, Matsuda H and Iwasa Y 1997 *J. Theor. Biol.* **184** 65
- [13] Szabó G and Töke C 1998 *Phys. Rev. E* **58** 69
- [14] Brauchli K, Killingback T and Doebeli M 1999 *J. Theor. Biol.* **200** 405
- [15] Szabó G, Antal T, Szabó P and Droz M 2000 *Phys. Rev. E* **62** 1095
- [16] Hauert C 2002 *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **12** 1531
- [17] Trivers R L 1971 *Q. Rev. Biol.* **46** 35
- [18] Alexander R D 1987 *The Biology of Moral Systems* (New York: de Gruyter)
- [19] Nowak M A and Sigmund K 1998 *Nature* **393** 573
- [20] Wedekind C and Milinski M 2000 *Science* **288** 850
- [21] Hauert C, De Monte S, Hofbauer J and Sigmund K 2002 *Science* **296** 1129
- [22] Szabó G and Hauert C 2002 *Phys. Rev. Lett.* **89** 118101
- [23] Szabó G and Hauert C 2002 *Phys. Rev. E* **66** 062903
- [24] Semmann D, Krambeck H J and Milinski M 2003 *Nature* **425** 390
- [25] Traulsen A, Röhl T and Schuster H G 2005 *Phys. Rev. Lett.* **93** 028701
- [26] Foster D and Young P 1990 *Theor. Popul. Biol.* **38** 219
- [27] Cabrales A 2000 *Int. Econ. Rev.* **41** 451
- [28] Szabó G, Vukov J and Szolnoki A 2005 *Phys. Rev. E* **72** 047107
- [29] Broadbent S R and Hammerlsley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
- [30] Janssen H K 1981 *Z. Phys. B* **42** 151
- [31] Grassberger P 1982 *Z. Phys. B* **47** 365
- [32] Pikovsky A S and Kurths J 1997 *Phys. Rev. Lett.* **78** 775
- [33] Perc M 2005 *Phys. Rev. E* **72** 016207  
Perc M 2005 *Europhys. Lett.* **72** 712
- [34] Huberman B A and Glance N S 1993 *Proc. Natl. Acad. Sci. USA* **90** 7716
- [35] Gammaitoni L, Hänggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
- [36] Perc M and Marhl M 2005 *Phys. Rev. E* **71** 026229
- [37] Szabó G and Szolnoki A 1996 *Phys. Rev. E* **53** 2196
- [38] Hauert C and Szabó G 2005 *Am. J. Phys.* **73** 405
- [39] Hu G, Ditzinger T, Ning C Z and Haken H 1993 *Phys. Rev. Lett.* **71** 807

- [40] Rappel W J and Strogatz S H 1994 *Phys. Rev. E* **50** 3249
- [41] Wio H S 1995 *Phys. Rev. E* **54** R3075
- [42] Han S K, Yim T G, Postnov D E and Sosnovtseva O V 1999 *Phys. Rev. Lett.* **83** 1771
- [43] Neiman A, Schimansky-Geier L, Cornell-Bell A and Moss F 1999 *Phys. Rev. Lett.* **83** 4896
- [44] Zhou C, Kurths J and Hu B 2001 *Phys. Rev. Lett.* **87** 098101
- [45] Carrillo O, Santos M A, García-Ojalvo J and Sancho J M 2004 *Europhys. Lett.* **65** 452
- [46] Perc M 2005 *Chem. Phys. Lett.* **410** 49
- [47] Perc M 2005 *New J. Phys.* **7** 252