

## Minimal model for spatial coherence resonance

Matjaž Perc\* and Marko Marhl

*Department of Physics, Faculty of Education, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia*

(Received 28 November 2005; published 5 June 2006)

We show that a planar medium, locally modeled by a simple one-dimensional excitable system with a piece-wise linear potential, can serve as a minimal model for spatial coherence resonance. Via an analytical treatment of the spatially extended system, we derive the dependence of the resonant wave number on several crucial system parameters, ranging from the diffusion coefficient to the local excursion time of constitutive excitable units. Thus, we provide vital insights into mechanisms that enable the emergence of exclusively noise-induced spatial periodicity in excitable media.

DOI: [10.1103/PhysRevE.73.066205](https://doi.org/10.1103/PhysRevE.73.066205)

PACS number(s): 05.45.-a, 05.40.-a, 89.75.Kd

### I. INTRODUCTION

It is a well-established fact that noise can have constructive effects on the dynamics of nonlinear systems [1]. Probably the most famous phenomenon related to this rather counterintuitive fact is the stochastic resonance [2], which stands for the resonant noisy enhancement of the correlation between the system's response and a weak external stimulus [3–11]. Fascinatingly, noise can play an ordering role even in the absence of additional external signals, whereby the established term describing the phenomenon is coherence resonance [12–15].

Following advances in the study of effects of noise on the dynamics of temporal systems, recent years have witnessed a big increase in scientific literature devoted to the analysis of noise-induced phenomena also in spatially extended systems [16]. The so-called spatiotemporal stochastic resonance has been first reported in Ref. [17], while spatial coherence resonance was introduced in Ref. [18] for systems near pattern forming instabilities and in Ref. [19] for excitable media. Moreover, there exist studies reporting noise-induced spiral growth and enhancement of spatiotemporal order [20–25], noise-sustained coherence of space-time clusters and self-organized criticality [26], noise-enhanced and -induced excitability [27,28], persistency of noise-induced spatial periodicity [29], noise-induced propagation of harmonic signals [30], noise-sustained and controlled synchronization [31], as well as spatial decoherence due to small-world connectivity [32] in spatially extended systems.

In the present study, we propose a simple one-dimensional excitable system with a piece-wise linear potential to be used as the constitutive unit of a two-dimensional excitable media yielding, due to its simplicity, a minimal model for spatial coherence resonance. We show that additive spatiotemporal random perturbations with an appropriate intensity are able to extract a particular spatial frequency of the studied media in a resonant manner, thus confirming the appropriateness of the model for the designated role. Importantly, due to the simple kinetics of the constitutive excitable units, the spatially extended system can be analyzed analytically. In particular, we succeed in explicitly linking the resonant

noise induced wave number  $k_{\max}$  with the parameters determining the local system dynamics, as well as the diffusion coefficient  $D$ . Thus, we derive analytically the conjecture  $k_{\max} \propto 1/\sqrt{D}$ , which we have reported previously for the excitable media with FitzHugh-Nagumo local dynamics solely on the basis of a quantitative analysis [19], as well as provide an explanation for the emergence of periodic spatial waves in excitable media out of noise. Finally, we show that the numerical results are in excellent agreement with the theoretical predictions, thus validating our analytical treatment.

The paper is structured as follows. Section II is devoted to the description of the mathematical model and its main “local” characteristics. In Secs. III and IV evidence for the spatial coherence resonance and the analytical treatment is presented, respectively. In the last section, we summarize the results and outline possible applications of our findings.

### II. MATHEMATICAL MODEL

Since the essential role of slow and fast dynamics for coherence resonance in excitable systems is well documented and established [13–15], it is reasonable to demand that a mathematical model should incorporate both features if it is to be used successfully for demonstrating such phenomena. As already established by Pradines *et al.* [15], both ingredients are readily incorporated in a simple one-dimensional model with a doubly piece-wise linear potential given by

$$\frac{du}{dt} = f(u) = (1-a)\Theta(u_c - u) + b\Theta(u - u_c), \quad (1)$$

where  $0 \leq u < 2\pi$  is the phase of the system,  $\Theta$  is the Heaviside function,  $u_c > 0$  is the firing threshold, while parameters  $a > 1$  and  $b > 0$  determine the kinetics of the system for  $u < u_c$  and  $u > u_c$ , respectively. For the above parameter values the system has a single excitable steady state at  $u=0$ . Small perturbations of the excitable steady state evoke large-amplitude spikes ( $u=2\pi$ ), provided  $u$  exceeds  $u_c$ . Importantly, large-amplitude spikes occur solely due to the noise-induced threshold crossing events of variable  $u$  in the state space, and thus should not be attributed to the proximity of system parameters to the oscillatory dynamical regime that emerges for  $a < 1$ .

\*Electronic address: [matjaz.perc@uni-mb.si](mailto:matjaz.perc@uni-mb.si)

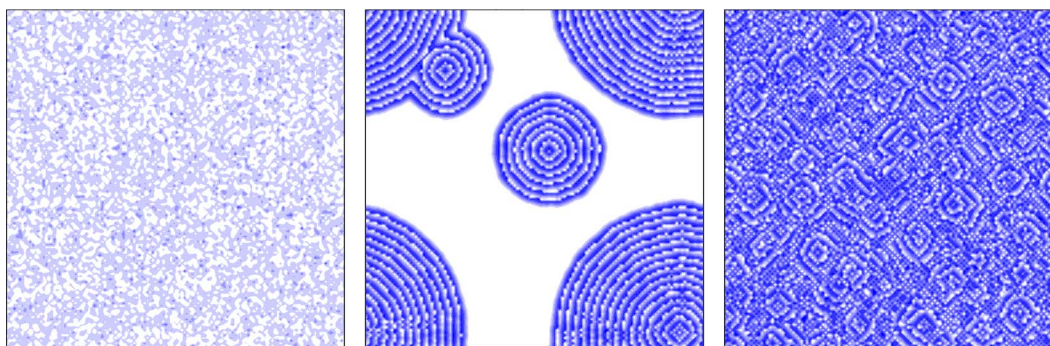


FIG. 1. (Color online) Characteristic snapshots of the spatial profile of  $u$  for  $\sigma=0.027$  (left),  $\sigma=0.035$  (middle), and  $\sigma=0.040$  (right). All figures are drawn using a linear color profile white marking 0.0 and blue  $2\pi$  values of  $u$ , whereby the color scale in the left panel is divided by a factor of 100 to enable visualization of small-amplitude excitations.

The above one-dimensional model is used as the constitutive unit for the spatially extended system, which we propose as the minimal model for spatial coherence resonance. The studied spatially extended system takes the form

$$\frac{du_{ij}}{dt} = f(u_{ij}) + D\nabla^2 u_{ij} + \xi_{ij}(t), \quad (2)$$

where  $u_{ij}$  is considered as a dimensionless two-dimensional scalar field on a discrete  $n \times n$  square lattice with mesh size  $\Delta x=1$ .  $\xi_{ij}(t)$  is temporally and spatially white additive Gaussian noise with zero mean satisfying the correlation  $\langle \xi_{ij}(t)\xi_{op}(t') \rangle = \sigma^2 \delta(t-t')\delta_{io}\delta_{jp}/\Delta x^2$ , where  $\sigma/\Delta x$  is the standard deviation of the noise in a discrete space [16]. The Laplacian  $D\nabla^2 u_{ij}$ ,  $D$  being the diffusion coefficient, is incorporated into the numerical scheme via a five-point finite-difference formula as described by Barkley [33], using periodic boundary conditions. System parameters used in subsequent calculations are  $a=1.05$ ,  $b=8.0$ ,  $u_c=\pi/30$ ,  $D=0.32$ , and  $n=128$ . Moreover, the system is initiated from steady state excitable conditions  $u_{ij}=0$  for  $\forall i,j$ .

### III. SPATIAL COHERENCE RESONANCE

In what follows, we will systematically analyze effects of different  $\sigma$  on the spatial dynamics of the media under study. We start by visually inspecting characteristic spatial profiles of  $u_{ij}$  obtained for three different values of  $\sigma$ . Results are presented in Fig. 1. It is evident that there exists an interme-

diante value of  $\sigma$ , for which coherent pattern formation in the media is resonantly pronounced, yielding well-ordered spiral waves in the spatial profile of  $u_{ij}$ , as presented in the middle panel of Fig. 1. On the other hand, small  $\sigma$  are unable to excite the system strong enough to evoke any particular spatial dynamics in the media, while for larger  $\sigma$  the pattern formation becomes somewhat violent so that the spatial profile again lacks any visible structure or order.

To quantify effects of various  $\sigma$  on the spatial scale of the studied system, we calculate the spatial structure function according to the equation

$$P(k_x, k_y) = \langle H^2(k_x, k_y) \rangle, \quad (3)$$

where  $H(k_x, k_y)$  is the spatial Fourier transform of the  $u$  field at a particular  $t$  and  $\langle \dots \rangle$  is the ensemble average over noise realizations. Figure 2 shows the results. As anticipated, small  $\sigma$  are unable to induce any particular spatial periodicity in the media, and thus  $P(\vec{k})$  is completely flat. On the other hand, near optimal  $\sigma$  clearly enhance a particular spatial scale of the media, which is indicated by the well-expressed circularly symmetric ring clearly visible in the middle panel of Fig. 2. For somewhat larger  $\sigma$ , noisy excitations start to indent ordered spatial waves in a random fashion, which disrupts the resonantly pronounced wave number. Note that the circularly symmetric ring in the rightmost panel of Fig. 2 is blurred in comparison to the middle panel.

To study results presented in Fig. 2 quantitatively, we exploit the circular symmetry of  $P(\vec{k})$  as proposed in Ref. [18].

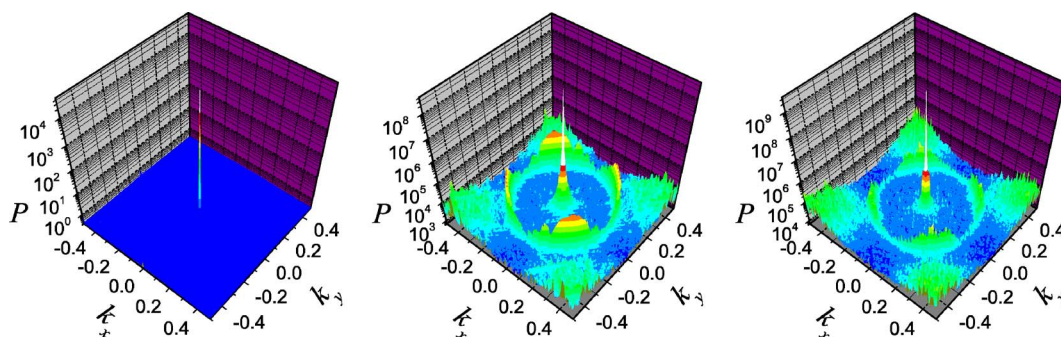


FIG. 2. (Color online) Spatial structure functions of  $u_{ij}$  obtained for the same noise levels as used in Fig. 1.

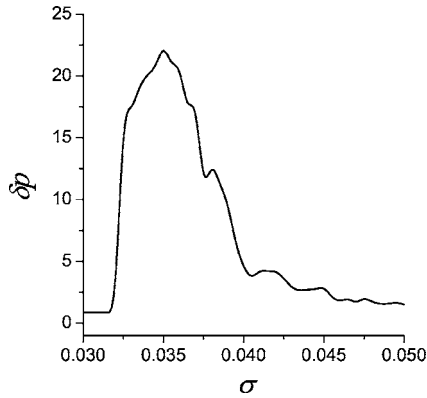


FIG. 3. Spatial coherence resonance in the studied system.

In particular, we calculate the circular average of the structure function according to the equation

$$p(k) = \int_{\Omega_k} P(\vec{k}) d\Omega_k, \quad (4)$$

where  $\Omega_k$  is a circular shell of radius  $k=|\vec{k}|$ . To quantify the ability of each particular noise level to extract the characteristic spatial periodicity in the system more precisely, we calculate the quantity  $\delta p = p(k_{\max})/\bar{p}$ , where  $\bar{p} = [p(k_{\max} - \Delta k_a) + p(k_{\max} + \Delta k_b)]/2$  is an approximation for the level of background fluctuations in the system. Thereby,  $\Delta k_a$  and  $\Delta k_b$  mark the first local minima occurring to the left and right side of  $p(k_{\max})$  respectively, hence determining the width of the peak around the resonantly pronounced wave number. Thus,  $\delta p$  measures the normalized height of the peak at  $k_{\max}$  for each particular  $\sigma$ . Figure 3 shows how  $\delta p$  varies with  $\sigma$ . It is evident that there exists an optimal level of additive noise for which the peak of the circularly averaged structure function at  $k_{\max}$  is best resolved, thus indicating the existence of spatial coherence resonance in the studied excitable media.

Importantly, since the dynamics of the media under study is qualitatively identical as in Ref. [19], the same explanation for the reported spatial coherence resonance applies. In particular, both the FitzHugh-Nagumo system [19] and the presently studied model have a stable excitable node, which has a state-space-based excitability threshold. If the steady state is perturbed strongly enough, both systems exhibit a large amplitude excitation in the state space, before quickly resetting onto the excitable node. The large-amplitude excitation phase is in both cases robust to noise, meaning that excitatory spikes cannot be overridden by the same weak noise that suffices to evoke an excitation. Thus, local excitations can propagate through the media in a persistent and robust manner. We argue that if a constitutive unit of a diffusively coupled excitable media satisfies these conditions, the latter qualifies for the observation of spatial coherence resonance, as reported above.

In Ref. [19] it was argued that the noise-robust excursion time  $t_c$  that is characteristic for the local dynamics of excitable units, together with the spread rate proportional to  $\sqrt{D}$  with which excitations propagate through the media, consti-

tute an inherent spatial scale that can be resonantly enhanced by additive random spatiotemporal perturbations, thus enabling spatial coherence resonance in the system. By exploiting these assumptions, we have shown via a quantitative analysis that  $k_{\max} \propto 1/\sqrt{t_c D}$ . Due to the minimalistic form of the presently studied model, this conjecture can be derived analytically. Moreover, below we present an explanation for the emergence of noise-induced periodic spatial waves in excitable media, thus explaining the essence of the reported spatial coherence resonance via a rigorous mathematical analysis.

#### IV. ANALYTICAL TREATMENT

Prior to investigating the dynamics of the spatially extended system, we first outline some obvious properties of a single excitable unit with respect to parameters  $a$ ,  $b$ , and  $u_c$  that determine its dynamics. To excite a given unit from its steady state at  $u=0$  the variable has to exceed the threshold  $u=u_c$  in the state space. In doing so, however, the external forcing has to overcome the system's internal resistance given by  $du/dt=1-a$ , which imposes a tendency towards  $u=0$  [note that  $1-a < 0$  in Eq. (1)]. Once a given unit exceeds  $u=u_c$ , its dynamics is determined by the fast kinetics  $u=bt$ .

To capture the essence of the noise-induced dynamics of the spatially extended system we, in the following, consider only two coupled units. Specifically, we study the influence of an excited unit on its not-yet excited ( $u < u_c$ ) neighbor. The influence of the excited neighbor on the dynamics of the quiescent unit can be approximated by the flux  $D(bt-u)$ . Thus, the dynamics of a quiescent unit, coupled with an already excited unit, can be described by the equation

$$\frac{du}{dt} = 1 - a + D(bt - u). \quad (5)$$

Integrating Eq. (5) for the initial condition  $u|_{t=0}=0$  gives the implicit equation for time  $t_c$  in which the critical value  $u=u_c$  is reached. The equation reads

$$ce^{-Dt_c} + bt_c = u_c + c, \quad (6)$$

where  $c=(a+b-1)/D$ . In order to obtain an explicit expression for  $t_c$  we simplify Eq. (6) by applying the approximation  $e^{-Dt_c} \approx 1 - Dt_c + D^2 t_c^2/2$ . By retaining the physically relevant positive solution and considering also that  $b \gg a-1$ , we obtain

$$t_c \approx \frac{a-1 + \sqrt{2bDu_c}}{bD}. \quad (7)$$

The estimated time  $t_c$ , in which a given unit starting from  $u=0$  reaches the threshold  $u=u_c$ , represents the time in which the excitation is transmitted from the excited to the neighboring quiescent unit. The propagation of the spatial wave therefore strongly depends on  $t_c$ . Actually, in case of periodic waves, the next wave front always appears after an integer multiple of  $t_c$ . Thus, the wavelength  $\lambda$ , representing the distance between two neighbouring wave fronts, is determined by  $\lambda=(t_0/t_c)\Delta x$ , where  $t_0$  is the oscillation period of

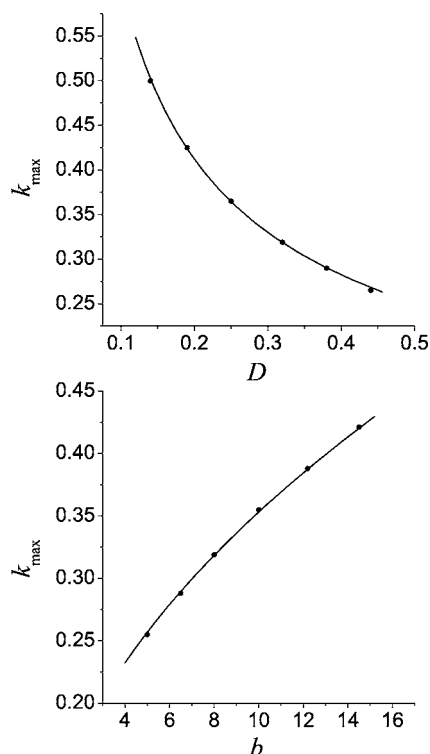


FIG. 4. Resonant wave number in dependence on  $D$  (top) and  $b$  (bottom). Dots indicate numerically obtained values, whereas solid lines indicate the predicted dependence given by Eq. (9).

every single unit in the spatially extended system. Acknowledging the fact that the firing period is practically equal to the noise-robust excursion time  $t_e$ , the spatial period of the waves  $\lambda$  is determined by

$$\lambda = \frac{t_e}{t_c} \Delta x. \quad (8)$$

Since  $b \gg a - 1$  and  $u_c \ll 2\pi$  the excursion time of the system given by Eq. (1) can be approximated analytically by  $t_e \approx 2\pi/b$ . By inserting this relation,  $\Delta x = 1$ , and Eq. (7) into Eq. (8), and expressing the spatial dynamics in terms of the inherent wave number  $k_{\max} = 1/\lambda$ , we finally obtain the equation

$$k_{\max} \approx \frac{a - 1 + \sqrt{2bDu_c}}{2\pi D}, \quad (9)$$

which links the noise-induced spatial periodicity with all parameters determining the dynamics of a single unit of the spatially extended system as well as the spatial coupling constant. It is evident that  $k_{\max} \propto 1/\sqrt{D}$  and  $k_{\max} \propto \sqrt{b} \propto 1/\sqrt{t_e}$ , thus confirming quantitatively based conjectures presented in

Ref. [19]. Moreover, the above treatment clearly reveals the mechanism behind the noise-induced spatial periodicity, and thus in turn explains the essence of spatial coherence resonance in excitable media.

To evaluate the accuracy of Eq. (9), we compare analytical predictions with values resulting from the numerical integration of Eq. (2). However, since the above analytical treatment was conducted without taking explicitly into account noise (note that we have just assumed that one unit is excited and the other one not), we first have to take into account nonzero values of  $\sigma$ . This is done simply by acknowledging the fact that nonzero  $\sigma$  decrease the effective excitability threshold in the state space of each spatial unit. Thus, when integrating Eq. (5) the initial state of variable  $u$  is not exactly zero, but in fact  $u|_{t=0} = \varepsilon > 0$ . For the near optimal  $\sigma$  resulting in maximal values of  $\delta p$ , we found that on average  $\varepsilon = 0.35u_c$ . This has the same effect as if  $u_c$  in Eq. (9) is replaced by the effective threshold  $0.65u_c$ , which is also the value we have used for evaluating the results presented in Fig. 4. It is evident that the analytically predicted values of  $k_{\max}$  obtained by Eq. (9) are in excellent agreement with the numerically calculated values for the near optimal  $\sigma$ , thus validating our above arguments and treatment.

## V. SUMMARY

We present a minimal model for spatial coherence resonance that is locally modeled by a simple one-dimensional excitable system with a doubly piece-wise linear potential. In particular, we show that additive spatiotemporally white Gaussian noise is able to extract an inherent spatial frequency of the studied media in a resonant manner. Thereby, no additional deterministic inputs were introduced to the system and the latter was initiated from steady state initial conditions. Due to the minimalistic form of the presently studied model, several important aspects of the noise-induced spatial dynamics can be analyzed analytically. We express the wave number of noise-induced waves as a function of all parameters determining the dynamics of a single unit of the spatially extended system as well as the diffusion coefficient. In doing so, we confirm conjectures of Ref. [19] in a more rigorous fashion, as well as provide an explanation for the emergence of periodic spatial waves in excitable media out of noise. Since excitability and noise appear to be present in various areas of science, ranging from chemistry, neurophysiology, cardiology to laser optics [16,34], we hope that the presented results will be of value to a broad readership. Moreover, since the simple dynamics of the studied system should also be fairly easily to mimic experimentally, we argue that the presented model is well suited for experimental investigations of the described phenomenon.

- [1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).
- [2] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).
- [3] J. K. Dougllass, L. A. Wilkens, E. Pantazelou, and F. Moss, *Nature* **365**, 337 (1993).
- [4] H. A. Braun, H. Wissing, K. Schäfer, and M. C. Hirsch, *Nature (London)* **367**, 270 (1994).
- [5] K. Wiesenfeld and F. Moss, *Nature (London)* **373**, 33 (1995).
- [6] R. P. Morse and E. F. Evans, *Nat. Med.* **2**, 928 (1996).
- [7] E. Simonotto, M. Riani, C. Seife, M. Roberts, J. Twitty, and F. Moss, *Phys. Rev. Lett.* **78**, 1186 (1997).
- [8] I. Hidaka, D. Nozaki, and Y. Yamamoto, *Phys. Rev. Lett.* **85**, 3740 (2000).
- [9] T. Mori and S. Kai, *Phys. Rev. Lett.* **88**, 218101 (2002).
- [10] E. Manjarrez, G. Rojas-Piloni, I. Méndez, and A. Flores, *J. Neurosci.* **23**, 1997 (2003).
- [11] K. Kitajo, D. Nozaki, L. M. Ward, and Y. Yamamoto, *Phys. Rev. Lett.* **90**, 218103 (2003).
- [12] G. Hu, T. Ditzinger, C. Z. Ning, and H. Haken, *Phys. Rev. Lett.* **71**, 807 (1993).
- [13] W. J. Rappel and S. H. Strogatz, *Phys. Rev. E* **50**, 3249 (1994).
- [14] A. S. Pikovsky and J. Kurths, *Phys. Rev. Lett.* **78**, 775 (1997).
- [15] J. R. Pradines, G. V. Osipov, and J. J. Collins, *Phys. Rev. E* **60**, 6407 (1999).
- [16] J. García-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems* (Springer, New York, 1999).
- [17] P. Jung and G. Mayer-Kress, *Phys. Rev. Lett.* **74**, 2130 (1995).
- [18] O. Carrillo, M. A. Santos, J. García-Ojalvo, and J. M. Sancho, *Europhys. Lett.* **65**, 452 (2004).
- [19] M. Perc, *Phys. Rev. E* **72**, 016207 (2005).
- [20] P. Jung and G. Mayer-Kress, *Chaos* **5**, 458 (1995).
- [21] P. Jung, A. Cornell-Bell, F. Moss, S. Kadar, J. Wang, and K. Showalter, *Chaos* **8**, 567 (1995).
- [22] J. García-Ojalvo and L. Schimansky-Geier, *Europhys. Lett.* **47**, 298 (1999).
- [23] H. Hempel, L. Schimansky-Geier, and J. García-Ojalvo, *Phys. Rev. Lett.* **82**, 3713 (1999).
- [24] S. Alonso, I. Sendiña-Nadal, V. Pérez-Muñuzuri, J. M. Sancho, and F. Sagués, *Phys. Rev. Lett.* **87**, 078302 (2001).
- [25] H. Busch and F. Kaiser, *Phys. Rev. E* **67**, 041105 (2003).
- [26] P. Jung, *Phys. Rev. Lett.* **78**, 1723 (1997).
- [27] J. García-Ojalvo, F. Sagués, J. M. Sancho, and L. Schimansky-Geier, *Phys. Rev. E* **65**, 011105 (2001).
- [28] E. Ullner, A. Zaikin, J. García-Ojalvo, and J. Kurths, *Phys. Rev. Lett.* **91**, 180601 (2003).
- [29] M. Perc, *Europhys. Lett.* **72**, 712 (2005).
- [30] A. A. Zaikin, J. García-Ojalvo, L. Schimansky-Geier, and J. Kurths, *Phys. Rev. Lett.* **88**, 010601 (2002).
- [31] C. S. Zhou and J. Kurths, *New J. Phys.* **7**, 18 (2005).
- [32] M. Perc, *New J. Phys.* **7**, 252 (2005).
- [33] D. Barkley, *Physica D* **49**, 61 (1991).
- [34] B. Lindner, J. García-Ojalvo, A. Neiman, and L. Schimansky-Geier, *Phys. Rep.* **392**, 321 (2004).