Transition from Gaussian to Levy distributions of stochastic payoff variations in the spatial prisoner's dilemma game

Matjaž Perc*

Department of Physics, Faculty of Education, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia

(Received 26 September 2006; published 2 February 2007)

We study the impact of stochastic payoff variations with different distributions on the evolution of cooperation in the spatial prisoner's dilemma game. We find that Gaussian-distributed payoff variations are most successful in promoting cooperation irrespective of the temptation to defect. In particular, the facilitative effect of noise on the evolution of cooperation decreases steadily as the frequency of rare events increases. Findings are explained via an analysis of local payoff ranking violations. The relevance of results for economics and sociology is discussed.

DOI: 10.1103/PhysRevE.75.022101 PACS number(s): 02.50.Le, 05.40.Ca, 87.23.Ge

According to the fundamental principles of Darwinian selection all individuals should act selfishly in order to maximize their fitness and assure best conditions for producing offspring. The famous "only the fittest survive" principle thus assumes an innate selfish drive that is inherently rooted in each individual, forcing it to act exclusively in its own good, thereby not paying any attention to the harm inflicted on the neighbor or the population. This unadorned scenario is concisely described by the classical well-mixed prisoner's dilemma game [1], where cooperators always die out. However, healthy and successful societies depend on individuals cooperating for the common good, even at the risk of personal loss. Thus, the discrepancy between theory and real-life experience as well as observations [2] dictates the need for more sophisticated theoretical approaches bridging the gap.

Several mechanisms and theoretical supplements to the classical prisoner's dilemma as well as other games have been proposed to describe the persistence of cooperative behavior. Spatial extensions [3], reciprocity [4], and strategic complexity [5] are well established as being potent promoters of cooperation. However, although very successful, these mechanisms still require certain conditions to be met in order for cooperative behavior to survive. Thus, it is often within the framework of these seminal theoretical approaches that additional or supplemental cooperation-facilitating mechanisms are sought in order to assure as precise a description of real-life scenarios as possible. Recently, a very promising avenue of research has proved to be the addition of stochasticity at some level of the game, thus resulting in a fruitful consolidation of physics and evolutionary game theory [6]. Stochastic gain in population dynamics has been reported in [7], while noise-induced cooperation promotion in the spatial prisoner's dilemma game has been presented in [8]. Small-world and other complex topologies of players on the spatial grid have also been identified as being relevant by the evolutionary process [9], as were the effects of finiteness in population size [10].

Presently, we study an important extension of noise-induced cooperation in the spatial prisoner's dilemma game by studying not only the impact of Gaussian noise, but also the effect of Lévy-distributed stochastic payoff variations on the evolution of cooperation. Lévy distributions differ from the Gaussian in that rare events occur more frequently, depending on the value of the exponent via which the tails of the Lévy distribution taper off in a power law manner [11]. Perhaps the most important and celebrated field where Lévy fluctuations have emerged as being widespread and relevant is economics [12]. Thus, since unpredictable Lévy-distributed payoff variations are plausible, the present study addresses a relevant problem. We find that Lévy-distributed payoff variations are less successful by maintaining cooperation among the players on the spatial grid than their Gaussian counterparts. More precisely, we find that Gaussian-distributed payoff variations are most successful in promoting cooperation irrespective of the temptation to defect, while the facilitative effect of noise on the evolution of cooperation decreases steadily as the frequency of rare events increases. Due to the introduction of stochasticity in the payoffs, local violations of the prisoner's dilemma payoff ranking occur more or less often depending on the intensity of noise. We find that the onset of these violations holds the key to understanding noise-induced cooperation promotion in general, as well as the deterioration of the facilitative effect as the Gaussian distribution of payoff variations changes towards the Lévy type.

In the following, we consider an evolutionary two-strategy prisoner's dilemma game with players located on vertices of a two-dimensional square lattice of size $n \times n$ with periodic boundary conditions. Each individual is allowed to interact only with its four nearest neighbors located to the north, south, east, and west, whereby self-interactions are excluded. Cooperators (C) and defectors (D) are initially uniformly distributed on the square lattice. A player P_g ($g = 1, \ldots, n \times n$) can change its strategy after each full iteration cycle of the game, whereby the performance of one randomly chosen nearest neighbor P_h is taken into account according to

*Address for correspondence: University of Maribor, Department of Physics, Faculty of Education, Koroška cesta 160, SI-2000 Maribor, Slovenia. FAX: +386 2 2518180. Electronic address: matjaz.perc@uni-mb.si

$$W[P_g \leftarrow P_h] = \frac{1}{1 + \exp[(S_g - S_h)/K]}, \qquad (1)$$

where $K=0.1$ is the uncertainty related to the strategy adoption process [6]. The cumulative payoffs of both players (S_g, S_h), acquired during each iteration cycle, are calculated in accordance with the payoff matrix

$$\begin{array}{c|cc} \frac{P_g}{P_h} & C & D \\ \hline C & \frac{1+\xi_g}{1+\xi_h} & \frac{1+r+\xi_g}{-r+\xi_h} \\ D & \frac{-r+\xi_g}{1+r+\xi_h} & \frac{0+\xi_g}{0+\xi_h} \end{array}. \qquad (2)$$

Two cooperators receive the reward $R=1$, and two defectors receive the punishment $P=0$, while a cooperator and defector receive the suckers payoff $S=-r$ and the temptation $T=1+r$, respectively, thus satisfying the prisoner's dilemma payoff ranking $T>R>P>S$ if the temptation to defect $r>0$. The presently applied additive noise (ξ_g, ξ_h) is defined by the characteristic function

$$\ln \phi(t) = -\sigma^{\alpha}|t|^{\alpha}[1 - i\beta \operatorname{sgn}(t)\tan(\pi\alpha/2)] + i\mu t, \qquad (3)$$

where $\alpha \in (1,2]$, $\beta \in [-1,1]$, and $\mu \in \mathbb{R}$. The corresponding α-stable distribution is $S_{\alpha}(\sigma, \beta, \mu)$ [11], whereby α defines the characteristic exponent determining the rate at which the tails of distributions taper off. If $1<\alpha \leqslant 2$, the mean of the distribution exists and equals μ. Moreover, σ is the scale parameter determining the width of the distribution, which by $\alpha=2$ is Gaussian with variance $2\sigma^2$. On the other hand, if $\alpha<2$, the variance is infinite, whereby the frequency of rare, or "big," events increases as α decreases [11]. Finally, parameter β determines the skewness of the distribution, which is leftward bound if $\beta<0$ and otherwise if $\beta>0$. The following analysis is constrained to the case where $\mu=0$, preserving the payoff ranking of the prisoner's dilemma over time among all interacting players, and $\beta=0$, resulting in symmetrical payoff variations with respect to positive and negative additions. Thus, the two main parameters are $\sigma \geqslant 0$ and $1<\alpha \leqslant 2$, determining the effective strength and tail behavior of the distribution of payoff variations, respectively.

The spatial prisoner's dilemma game studied is iterated forward in time using a synchronous update scheme, thus letting all individual interact pairwise with their four nearest neighbors. After every such iteration cycle of the game all players simultaneously update their strategy according to Eq. (1) and reset their cumulative payoffs to zero. For a large enough number of game iterations $(t \geqslant 10^5)$ and large system sizes $(n \geqslant 200)$, the average frequencies of cooperators F_C and defectors F_D approach an equilibrium value irrespective of the initial distribution of strategies, provided long enough discard times are taken into account. Figure 1 shows the severity of finite-size effects by a given σ and α for the defection temptation value $r=0.02$. By constant payoffs $(\sigma=0)$ cooperators are able to survive on the spatial grid only if r is smaller than a given threshold value, which for the presently applied game iteration scheme and player adoption rule equals $r_{\text{tr}}=0.006\,34$.

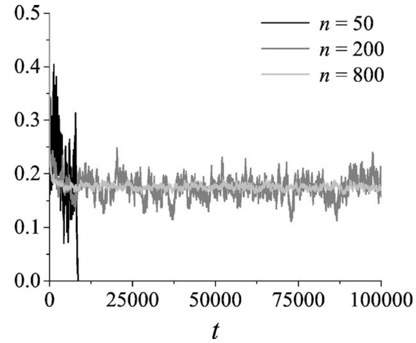

FIG. 1. Example of finite-size effects by $\sigma=0.15$ and $\alpha=1.4$. Lines depict current fractions of cooperators (F'_C) on the spatial grid. Clearly $n=50$ is too small as cooperators die out due to finiteness related stochasticity. By $n=200$ and $n=800$ values start to fluctuate around an equilibrium (after initial transients), whereby fluctuations are substantially smaller by the larger lattice. Nonetheless, if initial 3×10^4 values are discarded the average fraction of cooperators F_C differs absolutely only by ±0.002. Near extinction thresholds in Fig. 3 finite-size effects can be more severe, but the range of σ for which cooperators survive is accurate within $\pm3\%$ if $n=200$.

Next, we study the impact of different stochastic payoff variations on the equilibrium frequencies of cooperators on the spatial grid. Figure 2 shows characteristic snapshots of the spatial grid for three different values of α and $r=0.02$. Note that without the introduction of noisy payoff variations cooperators would go extinct. Remarkably, though, Gaussian-distributed payoff variations of appropriate σ are able to boost the fraction of cooperators to nearly 50%, as shown in the left panel of Fig. 2. However, as the Gaussian distribution is relaxed to follow the Lévy distribution by the same σ, obtained for all $\alpha<2$, the facilitative effect of noise on the cooperative strategy deteriorates substantially, as shown in the middle and right panels of Fig. 2.

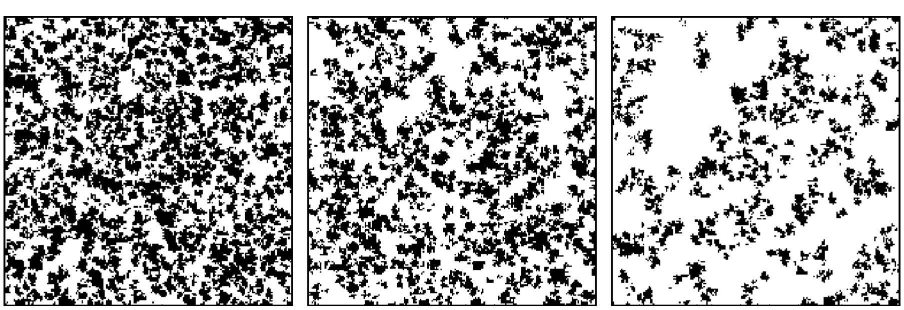

FIG. 2. Characteristic equilibrium spatial distributions of cooperators (black) and defectors (white) obtained by $\sigma=0.15$ and $\alpha=2.0$ ($F_C=0.48$, left panel), $\alpha=1.6$ ($F_C=0.36$, middle panel), and $\alpha=1.4$ ($F_C=0.18$, right panel) for the defection temptation value $r=0.02$. All panels are depicted on a 200×200 spatial grid.

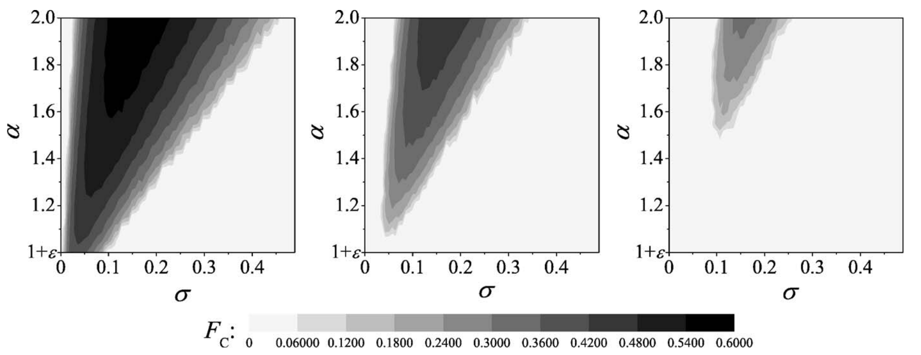

FIG. 3. Promotion of cooperation by stochastic payoff variations in dependence on σ and α for $r=0.01$ (left panel), $r=0.02$ (middle panel), and $r=0.03$ (right panel). In all cases $\varepsilon=0.001$ (see the y axis).

To study the trend outlined in Fig. 2 more precisely, we calculate F_C in dependence on σ and α for different $r>r_{\text{tr}}$, as shown in Fig. 3. It becomes instantly obvious that stochastic payoff variations are most successful in promoting cooperation if they follow a Gaussian distribution that is obtained by $\alpha=2$. As soon as $\alpha<2$ the facilitative effect deteriorates continuously irrespective of r. Not surprisingly, though, the overall facilitative effect of noise lessens as r increases from the left towards the right panel of Fig. 3 since the benefit of cooperation in comparison to possible losses decreases, and thus the mechanism of stochastic cooperation promotion can no longer compensate for this deterministic effect. It is also interesting to note that there always exist an intermediate σ for which the promotion of cooperation by noise is maximal. We recently argued that the phenomenon is conceptually identical to coherence resonance often reported within the framework of noise-driven dynamical systems [13].

Finally, it remains of interest to provide an explanation for the mechanism of stochastic cooperation promotion and for the deterioration of the effect as the Gaussian distribution of payoff variations changes towards the Lévy type. We argue that noise-induced payoff ranking violations of the prisoner's dilemma game hold the key to understanding. Since average additions to the payoffs of each player due to noise equal zero ($\langle \xi_g \rangle_{\text{time}}=\mu=0$ for $\forall\, g$), the payoff ranking $T>R>P>S$ is preserved on average over time. However, since the absolute magnitude of noise is allowed to exceed r or 1 locally—i.e., whenever two neighbors on the spatial grid interact—violations of the payoff ranking are possible at every instance of the game. We define two possible types of local payoff ranking violations that can occur whenever two individuals interact. First, let v_a denote the frequency of how often $T>R$ and $P>S$ rankings are violated. Note that both inequalities differ by r. Second, let v_b denote the frequency of how often the $R>P$ ranking is violated. Note that this inequality will be violated less often by a given σ than the former two since R and P differ by 1, which is substantially larger than r. We argue that the facilitative effect of noise on cooperation is directly related to v_a. In particular, if $v_a>0$, two cooperators might end up receiving a larger payoff each than a defector facing a cooperator. Also, a cooperator facing a defector might be better off than two defectors. These two facts obviously favor the cooperative strategy since they potentially nullify the advantage r defectors have over cooperators. On the other hand, this facilitative effect is limited by v_b. Namely, as $v_b>0$ two defectors might be better off than two cooperators, which again gives the winning edge to the defecting strategy, and hence results in a resonant dependence of cooperation fitness. Although being fairly simple, the described explanation outlines a general mechanism of cooperation promotion in the spatial prisoner's dilemma game.

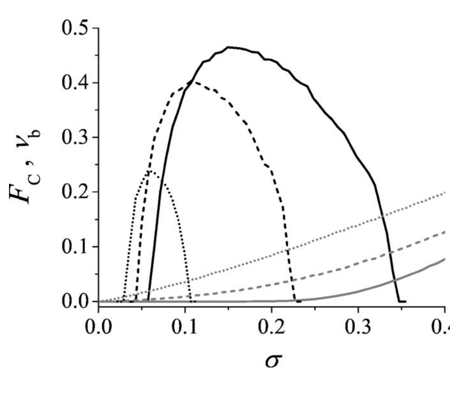

FIG. 4. Promotion of cooperation by $r=0.02$ for $\alpha=2.0$ (solid line), $\alpha=1.6$ (dashed line), and $\alpha=1.2$ (dotted line). Gray lines in corresponding line styles show the pertaining dependence of v_b on σ.

With the proposed explanation in mind, it is straightforward to see why Lévy-distributed payoff variations are less successful in promoting cooperation than Gaussian noise. In particular, as α decreases the frequency of rare events increases, and thus v_b grows faster in dependence on σ, which ultimately hinders the facilitative effect. More precisely, if $\alpha=2$, corresponding to Gaussian noise, v_b grows the slowest, yielding the best promotion of cooperation in comparison to Lévy-distributed disturbances, as shown in Fig. 4. In accordance with the proposed explanation, rare events by $\alpha<2$ also induce faster growing v_a, which results in cooperation facilitation already by smaller σ in comparison to Gaussian noise, but also narrows the overall range of σ in which cooperation promotion is still possible.

In sum, we show that Gaussian-distributed payoff variations are most successful in promoting cooperative behavior by a given effective strength of disturbances, σ. The facilitative effect of noise in general, as well as its deterioration by the transition from the Gaussian towards Lévy distributions, is attributed to local payoff ranking violations, which arguably hold the key to understanding the presented phenomena.

Lévy-distributed stochastic processes are common in economics [12], where they account for the statistical description of rare events (e.g., stock market breakdowns, sudden bankruptcies of large enterprises, etc.), which in reality occur far more often as one might have anticipated from the Gaussian distribution. Thus, they represent and important family of stochastic processes that apparently have wide-reaching consequences for the welfare of society. In future studies, it would be interesting to study the effects of stochastic payoff variations also in the framework of other games, such as, for example, the hawk-dove game [14], where subtle changes in payoff rankings have already been found to yield qualitatively different behavior on the spatial grid in comparison to the prisoner's dilemma game [15].

The author acknowledges support from the Slovenian Research Agency Z1-9629 and Programme No. P1-0055.

---

[1] R. Axelrod, *The Evolution of Cooperation* (Basic Books, New York, 1984); J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, England, 1998).
[2] G. S. Wilkinson, Nature (London) **308**, 181 (1984); R. M. Seyfarth and D. L. Cheney, *ibid.* **308**, 541 (1984); M. Milinski, *ibid.* **325**, 433 (1987); T. H. Clutton-Brock, M. J. O'Riain, P. N. Brotherton, D. Gaynor, R. Kansky, A. S. Griffin, and M. Manser, Science **284**, 1640 (1999).
[3] M. A. Nowak and R. M. May, Nature (London) **359**, 826 (1992); B. A. Huberman and N. S. Glance, Proc. Natl. Acad. Sci. U.S.A. **90**, 7716 (1993); M. A. Nowak and R. M. May, Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 35 (1993); M. A. Nowak, S. Bonhoeffer, and R. M. May, *ibid.* **4**, 33 (1994); G. Szabó, T. Antal, P. Szabó, and M. Droz, Phys. Rev. E **62**, 1095 (2000); C. Hauert, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 1531 (2002); G. Szabó and C. Hauert, Phys. Rev. Lett. **89**, 118101 (2002).
[4] R. L. Trivers, Q. Rev. Biol. **46**, 35 (1971); M. A. Nowak and K. Sigmund, Nature (London) **393**, 573 (1998); C. Wedekind and M. Milinski, Science **288**, 850 (2000).
[5] C. Hauert, S. De Monte, J. Hofbauer, and K. Sigmund, Science **296**, 1129 (2002); G. Szabó and C. Hauert, Phys. Rev. E **66**, 062903 (2002); D. Semmann, H. J. Krambeck, and M. Milinski, Nature (London) **425**, 390 (2003).
[6] G. Szabó and C. Töke, Phys. Rev. E **58**, 69 (1998); C. Hauert and G. Szabó, Am. J. Phys. **73**, 405 (2005).
[7] A. Traulsen, T. Röhl, and H. G. Schuster, Phys. Rev. Lett. **93**, 028701 (2004).
[8] G. Szabó, J. Vukov, and A. Szolnoki, Phys. Rev. E **72**, 047107 (2005); M. Perc, New J. Phys. **8**, 22 (2006).
[9] G. Abramson and M. Kuperman, Phys. Rev. E **63**, 030901(R) (2001); M. G. Zimmermann, V. M. Eguíluz, and M. S. Miguel, *ibid.* **69**, 065102(R) (2004); M. G. Zimmermann and V. M. Eguíluz, *ibid.* **72**, 056118 (2005); F. C. Santos and J. M. Pacheco, Phys. Rev. Lett. **95**, 098104 (2005); F. C. Santos, J. M. Pacheco, and T. Lenaerts, Proc. Natl. Acad. Sci. U.S.A. **103**, 3490 (2006); M. Perc, New J. Phys. **8**, 183 (2006); J. Vukov, G. Szabó, and A. Szolnoki, Phys. Rev. E **73**, 067103 (2006).
[10] A. Traulsen, J. C. Claussen, and C. Hauert, Phys. Rev. Lett. **95**, 238701 (2005); Phys. Rev. E **74**, 011901 (2006).
[11] L. Devroye, *Non-Uniform Random Variate Generation* (Springer, New York, 1986).
[12] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics* (Cambridge University Press, Cambridge, England, 2000).
[13] M. Perc and M. Marhl, New J. Phys. **8**, 142 (2006).
[14] M. Tomassini, L. Luthi, and M. Giacobini, Phys. Rev. E **73**, 016132 (2005).
[15] C. Hauert and M. Doebeli, Nature (London) **428**, 643 (2004).