Proximity to periodic windows in bifurcation diagrams as a gateway to coherence resonance in chaotic systems

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(Received 13 April 2007; published 24 September 2007)

We show that chaotic states situated in the proximity of periodic windows in bifurcation diagrams are eligible for the observation of coherence resonance. In particular, additive Gaussian noise of appropriate intensity can enhance the temporal order in such chaotic states in a resonant manner. Results obtained for the logistic map and the Lorenz equations suggest that the presented mechanism of coherence resonance is valid beyond particularities of individual systems. We attribute the findings to the increasing attraction of imminent periodic orbits and the ability of noise to anticipate their existence and use a modified wavelet analysis to support our arguments.

DOI: 10.1103/PhysRevE.76.037201 PACS number(s): 05.45.Ac, 05.40.Ca

When noise is introduced to nonlinear systems one can observe a variety of fascinating phenomena [1]. For example, the phenomenon of stochastic resonance occurs when an appropriate intensity of noise evokes the best correlation between a weak deterministic stimulus and the system's response [2]. This contradicts intuitive reasoning that suggests noise can only act destructive. Noteworthy, noise alone is also able to induce or enhance temporal order in the dynamics of a nonlinear system. The term coherence resonance has been suggested in [3] to describe noise-induced temporal order in an excitable FitzHugh-Nagumo model, yet similar phenomena have been observed already before [4].

Since the seminal works on stochastic and coherence resonance, excitability has been recognized as an important system property for a broad variety of noise-induced phenomena [5]. Similarly, proximities to special bifurcation points have also received substantial attention as being suitable dynamical states for the observation of stochastic and coherence resonances [6]. Related to the latter two phenomena are also system size [7] and diversity-induced [8] resonances. Noteworthy, a mushrooming field of research is also the study of effects of noise on spatially extended dynamical systems. In [9] a partial review of the field is given and under Ref. [10] some other works past the date of the previous referral are listed. Importantly however, this field is growing too fast to list here the most relevant contributions, and hence Refs. [9,10] should only be considered as guidance for the interested reader.

Intimately related to the content of the present Brief Report are studies focusing on the impact of noise in chaotic systems. Examples range from noise-induced order [11] and synchronization [12] to the stochastic resonance [13]. In [14] authors have shown that the Chua circuit operating in a chaotic regime with two co-existing stable chaotic attractors can exhibit coherence resonance. Specifically, the quantity of interest in [14] was the transition time between the two coexisting attractors. Liu and Lai have shown that the phenomenon of coherence resonance can also be observed in two coupled chaotic oscillators [15], whereby the quantity characterizing the difference between their dynamics exhibits a resonant dependence on the noise intensity. In [16] results presented in [15] were extended to several coupled units. Coherence resonance in coupled chaotic oscillators has also been studied by Zhan *et al.* [17], where it has been demonstrated that a periodic wave can be triggered under the influence of noise in parameter regions of synchronous chaos. Finally, we mention two interesting studies essentially reporting the opposite of coherence resonance in chaotic systems, namely noise-induced chaos [18], where it has been shown that noise can induce chaos in certain periodic states that are in the parametrical proximity of chaotic behavior.

Presently, we aim to extend the scope of coherence resonance in chaotic systems by showing that chaotic states in the proximity of periodic windows in bifurcation diagrams enable the observation of noise-enhanced temporal order, whereby the latter exhibits a resonant dependence on the intensity of noise. The phenomenon is demonstrated on the logistic map and the Lorenz equations [19], thus suggesting that the identified mechanism of coherence resonance is valid beyond particularities of individual systems such as discrete or continuous dynamics, or dimensionality. We argue that the proximity of chaotic states to periodic windows essentially acts as a near-bifurcation state [6], whereby noisy fluctuations can anticipate the dynamical behavior on the other side of the bifurcation in a resonant manner. More precisely, the imminent attraction of periodic orbits can be anticipated by noise, thus making the chaotic system visit the (yet unstable) regular state, in turn enhancing the order in the temporal output. We use a modified wavelet analysis [20], using as a wavelet the periodic solution emerging first in the periodic window, to support our arguments.

As announced, we use the logistic map and the Lorenz equations, whereby to both systems Gaussian noise is introduced additively. The noisy logistic map reads

$$w_{t+1} = \mu w_t(1 - w_t) + D\xi_t, \tag{1}$$

and the noisy Lorenz equations are

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$$\dot{x} = -\sigma(x-y) + D\xi_t^x, \tag{2}$$

$$\dot{y} = rx - y - xz + D\xi_t^y, \tag{3}$$

$$\dot{z} = xy - bz + D\xi_t^z. \tag{4}$$

In Eq. (1) t is an integer while in Eqs. (2)–(4) it represents continuous time. Moreover, D^2 is the variance of Gaussian noise, $\langle \xi_t \rangle = 0$ and $\langle \xi_t^i \xi_n^j \rangle = \delta_{tn}\delta_{ij}$, whereby $i,j \in (x,y,z)$. The bifurcation diagram of the logistic map has a large periodic window starting at $\mu = 3.83$, while for $b = 8/3$ and $\sigma = 10$ the Lorenz equations start to exhibit periodic behavior from $r = 99.5$ onwards [19]. In what follows we thus concentrate on the parameter regions $\mu < 3.83$ and $r < 99.5$, which places both systems in chaotic states that are close to periodic behavior, provided the distance to the boundary values denoting the advent of periodicity remains small. Since both systems are very well documented already in textbooks [19], we here omit the details about their deterministic dynamics and proceed with analyzing the impact of $D > 0$ on the temporal order of their output.

To quantify the temporal order we compute the normalized autocorrelation function

$$C(\tau) = \frac{\langle \tilde{q}_t \tilde{q}_{t+\tau} \rangle}{\langle \tilde{q}^2 \rangle}, \tag{5}$$

where $\tilde{q} = q - \langle q \rangle$ and q is either the temporal trace of w (logistic map) or of x (Lorenz equations). Finally, uniquely quantifying coherence resonance is the correlation time [3]

$$\tau_C = \int_0^\infty C^2(\tau) d\tau, \tag{6}$$

whereby the larger the value of τ_C the larger the temporal order in the studies series.

We start with the noisy logistic map. Figure 1 features results of quantification of temporal order for different μ and D. Clearly, noise is able to enhance the temporal order of the system's output for some μ in a resonant manner depending on D, thus marking the existence of coherence resonance in the chaotic logistic map. It is evident that as $\mu \to 3.83$ the maximally possible enhancement of temporal order due to noise increases, and moreover, that only chaotic states in the nearby vicinity of the periodic window warrant the observation of coherence resonance. Also, the optimal D for which the largest τ_C is obtained moves towards smaller D as $\mu \to 3.83$, which is in accordance with the expectation that smaller noise intensities have a larger impact if the system is closer to the periodic window. Note that qualitatively similar observations have been made for systems near bifurcation points [6], where stronger noise levels are required, and consequently an overall decrease of temporal order is observed, as the system moves further away from the bifurcation point.

Next, we consider the noisy Lorenz equations. Figure 2 shows how τ_C varies in dependence on D by different r. As in Fig. 1, it is evident that noise can enhance the temporal order in a chaotic state provided the latter is situated close to the periodic window. However, as this necessary condition is relaxed the coherence resonance first fades, moving towards higher D, and eventually vanishes completely. Due to considerable conceptual differences between the two studied models, results in Figs. 1 and 2 suggest that the presented mechanism of coherence resonance is quite general and should thus be easily transferable to other systems as well.

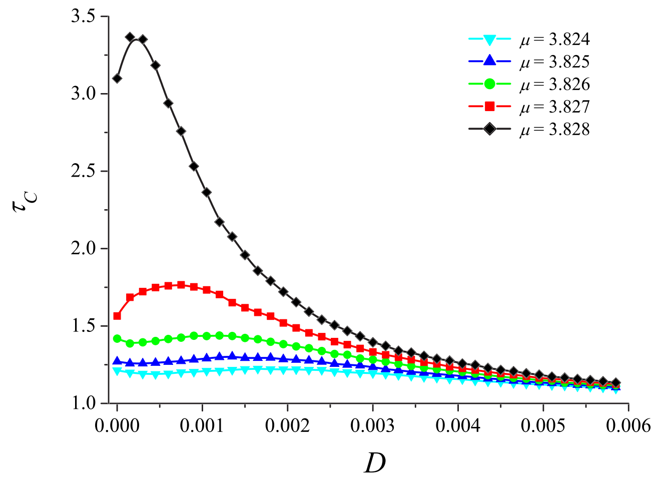

FIG. 1. (Color online) Coherence resonance in the chaotic logistic map near the periodic window starting at $\mu = 3.83$. The resonant dependence vanishes quickly as the system moves away from the periodic window. Points were obtained by averaging τ_C over 100 different realizations for each D. Lines are guides to the eye (also in subsequent figures).

Before we turn to explaining the mechanism behind the reported coherence resonance, we would like to note that the height of the resonance curve depends somewhat on the specific properties of chaos prior to the onset of periodicity. In particular, almost periodic, often called intermittent chaotic states (characterized by a positive yet close-to-zero maximal Lyapunov exponent), occurring in the proximity of some pe-

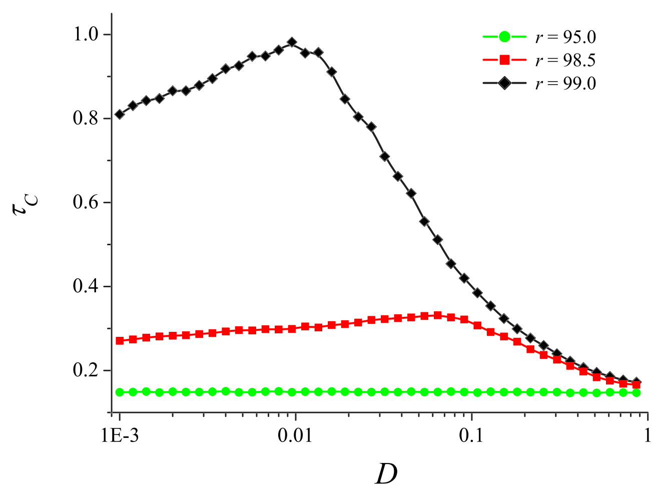

FIG. 2. (Color online) Coherence resonance in the chaotic Lorenz system near the periodic window starting at $r = 99.5$. Note that a resonant dependence on D can only be obtained in the proximity of the periodic window, whereas in the midst of chaos ($r = 95.0$) the impact of noise is negligible or at most destructive. Points were obtained by averaging τ_C over 100 different realizations for each D.

riodic windows, typically do not yield as convincing results as chaotic states with a highly positive maximal Lyapunov exponent. Aside from that the phenomenon is robust and can be observed also if the chaotic state is on the far side of the periodic window. Importantly however, by time-continuous systems the processor time required to obtain smooth and convincing resonance curves is quite large and numerical integration procedures of second order accuracy (e.g., Heun algorithm [9]) have to be used. In addition, averages over different realization by each D can prove superior to extremely long integration times, assuring better convergence control of τ_C.

In order to explain the mechanism behind the reported coherence resonance, we build on the fact that results in Figs. 1 and 2 show conceptual similarities with coherence resonances reported previously in proximities of bifurcation points [6], separating for example steady state and oscillatory solutions. We argue that presently the proximity to the periodic window plays essentially the same role, thus enabling noise to anticipate the ordered behavior in a resonant manner and enhance the temporal regularity of the dynamics. To support this argument we propose a simple procedure based on the wavelet analysis of temporal traces [20]. However, instead of using established orthonormal wavelets [21], we formally introduce the wavelet W_h being one oscillation period at the onset of periodicity, where $h=1,\ldots,p_{\max}$ counts the number of points it contains (e.g., for $\mu=3.83$ in the Logistic map $p_{\max}=3$). Next, we define the correlation function between the wavelet and the series

$$G(\tau)=K\sum_{h=1}^{p_{\max}}\tilde{q}_{\tau+h}\tilde{W}_h, \tag{7}$$

where $\tilde{q}=q-\langle q\rangle_\tau$ [$\langle q\rangle_\tau$ denoting the average of the segment of the series entering Eq. (7) by a particular τ], $\tilde{W}=W-\langle W\rangle$ and K is a normalization constant. Finally, the quantity determining the representation of the periodic orbit W_h in the series q is

$$\hat{\pi}=\int_0^\infty G^2(\tau)d\tau, \tag{8}$$

whereby for convenience we introduce the quantity $\pi=\hat{\pi}-\hat{\pi}_{D=0}$ which is simply a normalization with respect to the noise-free case. If $\pi>0$ as $D>0$, the correlation between the wavelet and the series of the system increases, thereby confirming that noise is able to anticipate the nearby periodic

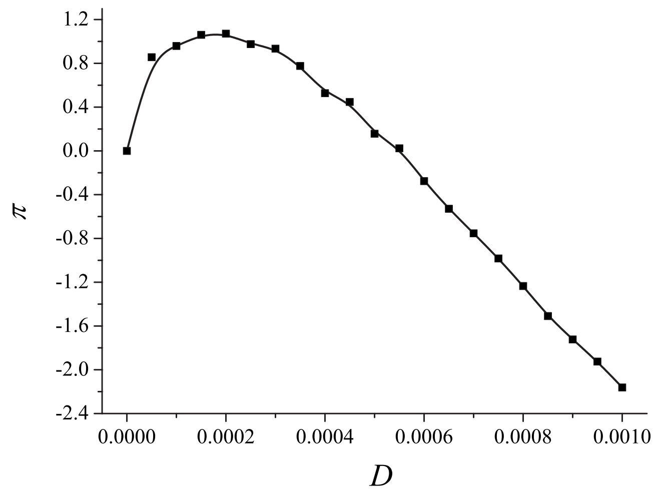

FIG. 3. Modified wavelet analysis of the chaotic logistic map at $\mu=3.828$, using as the wavelet the period-three periodic orbit at $\mu=3.83$. Noise is able to anticipate the imminent periodic behavior in a resonant manner depending on D.

behavior from the chaotic state. Results for the logistic map are presented in Fig. 3. Evidently, noise is indeed able to anticipate the imminent periodic behavior in a resonant manner, whereby the peak value of π is obtained by the same D as the peak of τ_C in Fig. 1. Results for the Lorenz system are qualitatively identical. This final result validates our proposed explanation, and sets the foundations for a new coherence resonance mechanism in chaotic states that is based on the proximity to periodic windows in bifurcation diagrams.

In sum, we present a mechanism warranting the observation of coherence resonance in chaotic systems. The necessary condition is the proximity of the chaotic state to a periodic window in the system's bifurcation diagram. The dynamics of such systems is similar to the dynamics caused by proximity to special bifurcation points in already familiar settings of stochastic and coherence resonance [6]. Presented results suggest that the phenomenon is largely independent of particularities of individual systems, and should thus be readily observed in other theoretical, as well as hopefully also experimental, setups. Our theory could prove useful for enhancing signal processing and detection [22] in chaotic states in systems that operate close to periodic behavior.

M.P. acknowledges support from the Slovenian Research Agency (Grant No. Z1-9629).

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