

Stochastic resonance on weakly paced scale-free networks

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We study the impact of additive Gaussian noise and weak periodic forcing on the dynamics of a scale-free network of bistable overdamped oscillators. The periodic forcing is introduced to a single oscillator and therefore acts as a pacemaker trying to impose its rhythm on the whole ensemble. We show that an intermediate intensity of temporally and spatially uncorrelated noise is able to optimally assist the pacemaker in achieving this goal, thus providing evidence for stochastic resonance on weakly paced scale-free networks. Because of the inherent degree inhomogeneity of individual oscillators forming the scale-free network, the placement of the pacemaker within the network is thereby crucial. As two extremes, we consider separately the introduction of the pacemaker to the oscillator with the highest degree and to one of the oscillators having the lowest degree. In both cases the coupling strength plays a crucial role, since it determines to what extent the whole network will follow the pacemaker on the expense of a weaker correlation between the pacemaker and the units that are directly linked with the paced oscillator. Higher coupling strengths facilitate the global outreach of the pacemaker, but require higher noise intensities for the optimal response. In contrast, lower coupling strengths and comparatively low noise intensities localize the optimal response to immediate neighbors of the paced oscillator. If the pacemaker is introduced to the main hub, the transition between the locally and globally optimal responses is characterized by a double resonance that postulates the existence of an optimal coupling strength for the transmission of weak rhythmic activity across scale-free networks. We corroborate the importance of the inhomogeneous structure of scale-free networks by additionally considering regular networks of oscillators with different degrees of coupling.

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I. INTRODUCTION

Stochastic resonance made its debut in science as a possible mechanism for the recurring occurrence of ice ages, and has since left its mark in an unprecedented number of different fields of research [1]. Still, however, the number of research works devoted to this fascinating phenomenon increases rapidly. Most commonly, stochastic resonance refers to the phenomenon when an appropriate intensity of noise evokes the best correlation between a weak periodic stimulus and the response of a nonlinear system. The fact that noise can constructively affect the functioning of any type of system contradicts intuitive reasoning, and indeed some form of nonlinearity is essential for the latter goal to be achieved [2]. It is noteworthy that closely related to the phenomenon of stochastic resonance is also the so-called coherence resonance [3], where noise alone suffices to induce a coherent response of the system. Although studies on stochastic resonance and other possible constructive effects of noise were initially focused mainly on single-unit systems, the scope shifted rather quickly to coupled arrays [4], where it has been discovered that the spatiality may additionally broaden the scope of stochastic [5,6] and coherence [7] resonance. Also, in this respect, two-dimensional media received substantial attention in the past [8], and recent advances have been comprehensively reviewed in [9].

In recent years, parallel with the expanding body of literature on the effects of noise on isolated and spatially ex-

tended nonlinear dynamical systems, the subject became increasingly imbued with the field of complex networks [10]. More precisely, the initially dominating nearest-neighbor interactions were replaced by more complex topologies such as small-world [11] and scale-free [12] networks. Rightfully so, one may argue, as such networks appear to be excellent for modeling interactions amongst units of complex systems, whereby examples include social networks [13], scientific-collaboration networks [14], food webs [15], computer networks [16], and neural networks [17]. Both stochastic [18] and coherence [19] resonance phenomena have already been studied in networks with small-world topology, and, in general, it has been reported that the introduction of shortcut links between randomly chosen units may increase the order of the dynamics, whereby the ordering effect depends largely on the coupling strength and the fraction of rewired links. Moreover, pattern formation and spatial order of spiral waves in media with small-world connections have also been studied [20], as were regularization effects of complex topologies and their ability to suppress spatiotemporal chaos [21] or induce bursting oscillations [22]. The idea of stochastic resonance has also been applied to opinion formation models [23], where the role of small-world topology, combined with external periodic modulation representing a so-called “fashion wave,” has been studied. Somewhat more closely related to the subject of the present work, recently an interesting study about the amplification of weak signals in scale-free networks of bistable oscillators has been published [24], and slightly earlier insightful findings regarding the synchronization on complex networks have been presented [25]. Directly linked with the current paper are two recent studies elaborating on the stochastic resonance phenomenon in coupled

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threshold elements [26] and the Ising model [27] on the Barabási-Albert network [12]. Our goal at present is to extend the subject by studying the stochastic resonance phenomenon on scale-free networks in the presence of localized weak rhythmic activity only.

We consider as the model the Barabási-Albert [12] scale-free network generated via the celebrated mechanism of growth and preferential attachment and populate it with bistable overdamped oscillators of which only a single one is subjected to subthreshold periodic forcing. The locally introduced periodic forcing thus acts as a pacemaker on the whole network. Pacemakers are important units of larger systems that dictate to their near and distant neighbors the operating rhythm or pace, and so guide the functioning of the whole ensemble. Most prominent systems with pacemakers, which, however, do not necessarily obey the principles of bistable dynamics or incorporate features characteristic of scale-free networks, are those from real life and include the human heart [28] and arteries and arterioles [29], as well as larger cells like eggs [30]. In accordance with their importance, several studies were already devoted to studying pacemaker impacts on excitable systems [31] and networks with small-world topology [32], as well as on propagation and refraction of chemical waves [33]. Presently, the introduction of a pacemaker serves to identify crucial influences of the scale-free topology on the phenomenon of stochastic resonance. In particular, due to the inherent degree inhomogeneity of individual oscillators forming the scale-free network, the placement of the pacemaker within the network is crucial; i.e., differences in the system's response can be expected if the pacemaker is introduced to a unit with a higher or lower degree. To address this, we consider as two extremes the introduction of the pacemaker to the oscillator with the highest degree (main hub) and to one of the oscillators having the lowest degree. We find that, irrespective of which options is chosen, the coupling strength plays a crucial role since it determines both quantitative and qualitative aspects of the reported stochastic resonance. More precisely, while higher coupling strengths ensure that the outreach of the pacemaker extends across the whole network, thus warranting a globally optimal response of the system, lower coupling strengths localize the optimal response to immediate neighbors of the directly paced oscillator. However, while the locally optimal response requires fairly low noise intensities, the globally optimal response relies on a substantially higher stochastic component. If the main hub hosts the pacemaker the latter fact results in a doubly resonant response as the noise intensity increases, whereby the first peak occurs when only the immediate neighbors of the directly paced oscillator are optimally correlated with the pacemaker, and the second peak occurs when the whole network is resonantly fine tuned. This doubly resonant response is virtually absent when the pacemaker is introduced to an oscillator with the lowest degree within the network since its immediate neighborhood is too small for the observation of a well-expressed locally optimal response. Finally, we show that the double-resonant response results in the existence of an optimal coupling strength for the noise-induced transmission of weak rhythmic activity across a scale-free network; in particular by warranting the best compromise between the locally and globally optimal

response. Findings obtained on scale-free networks are contrasted with results obtained by using homogeneous regular networks of bistable overdamped oscillators. We find that on such networks the locally optimal response is absent, or at least cannot be distinguished from the globally optimal response irrespective of the degree of coupled oscillators and the coupling strength, which corroborates the importance of the highly inhomogeneous structure of the scale-free network and its unique impact on the phenomenon of stochastic resonance.

The remainder of this paper is structured as follows. In Sec. II we introduce the model and other mathematical methods presently in use. Results are presented in Sec. III, and in the last section we summarize our findings.

II. MATHEMATICAL MODEL AND SETUP

The model to be used presently consists of noisy bistable overdamped oscillators, governed by Langevin equations of the form

$$\frac{dx_i}{dt} = x_i - x_i^3 + \sum_j \varepsilon_{ij}(x_j - x_i) + \sqrt{2D}\xi_i(t), \quad (1)$$

where ε_{ij} is the coupling strength between units i and j , and $2D$ is the variance of the Gaussian noise with zero mean and autocorrelation $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t')$. The dynamics of each oscillator is governed by two symmetric stable steady states centered around ± 1 that correspond to the minima of the perturbing potential energy function $V(x_i) = -x_i^2/2 + x_i^4/4$. Equation (1) arguably provides a paradigmatic setup for different scenarios of stochastic resonance, either via the classic setup, or via variations of system size [34] or diversity [35], hence implying that a fine tuning of D might optimize the response, i.e., switching between the two steady states, of the model in accordance with the frequency of a subthreshold periodic forcing.

To explore the possibility of stochastic resonance, we presently introduce a subthreshold pacemaker of the form $f_r(t) = A \cos(\omega t)$ to a single bistable oscillator $i=r$ that remains exposed to the periodic forcing during the whole simulation period. Throughout this study we use $A=0.3$ and $\omega = \pi/300$, which warrant that in the absence of noise ($D=0$) the pacemaker is subthreshold, meaning it cannot by itself induce transitions between the two stable steady states; not by the oscillator which is directly exposed and not by any other constitutive unit of the network. Instead, the bistable oscillator directly perturbed by the pacemaker (as well as those oscillators directly linked to it, but to a much lesser extent depending on the coupling strength) exhibits small-amplitude oscillations around the minimum of its potential with the frequency ω .

As the underlying interaction network we use the scale-free network generated via growth and preferential attachment as proposed by Barabási and Albert [12], comprising $N=200$ vertices. Each vertex corresponds to one noise-driven overdamped bistable oscillator. If oscillators i and j are connected then $\varepsilon_{ij} = \varepsilon_{ji} = \varepsilon$, but otherwise $\varepsilon_{ij} = \varepsilon_{ji} = 0$ and $\varepsilon_{ii} = 0$. Using the notation of [12], we start with $m_0=2$ con-

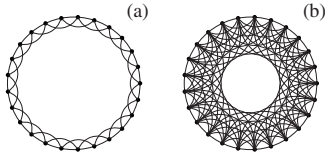


FIG. 1. Examples of regular network topologies considered with periodic boundary conditions. For clarity regarding the degree of each oscillator only 25 vertices are displayed in each panel. (a) Each vertex is connected to its $k_i=4$ nearest neighbors. (b) Each vertex is connected to its $k_i=10$ nearest neighbors. The extension to $k_i=36$, as used in Sec. III, is straightforward.

nected vertices, and subsequently every new vertex is attached to $m=2$ old vertices already present in the network, whereby the probability Π that a new vertex will be connected to vertex i depends on its degree k_i in accordance with $\Pi=k_i/\sum_j k_j$. This growth and preferential attachment scheme yields a network with an average degree $k_{av}=N^{-1}\sum k_i$ equaling 4, and a power-law degree distribution with the slope of the line equaling -2.9 on a double-logarithmic graph. Due to the large degree inhomogeneity of individual oscillators forming such a scale-free network, the placement of the pacemaker within the network is of crucial importance, and, as we will show below, may significantly influence the phenomenon of stochastic resonance. In order to address this, we consider the introduction of the pacemaker to the oscillator with the highest degree [$r=i(k_{max})$] and to one of the oscillators having the lowest degree [$r=i(k_{min})$]. The lowest degree is always $k_{min}=2$, whereas the highest degree varies for different generations of the network (note that the growth and preferential attachment procedure has inherently random components) and, for the currently considered network size comprising $N=200$ vertices, may occupy any value from the interval $k_{max} \in [20, 70]$, with the most probable value ≈ 35 (based on the statistics of 10^4 generated networks). Thus, in order to warrant statistical accuracy [especially if $r=i(k_{max})$] all results presented below were obtained as averages over 100 different realizations of the scale-free network. Importantly, although the dispersion of k_{max} is fairly large by $N=200$, the usage of much larger networks (with a somewhat smaller dispersion of k_{max}) would presently be inefficient because the subthreshold pacemaker could not influence a larger ensemble even in optimal conditions, and, moreover, the computer resources needed for an accurate simulation of such large ensembles would quickly exceed our options.

After presenting results obtained on scale-free networks, we also use, as the underlying interaction structure amongst bistable oscillators, homogeneous regular networks with different degrees, as exemplified in Fig. 1. More precisely, we consider regular ring networks with periodic boundary conditions comprising $N=200$ oscillators, each having degree $k_i=k_{av}=4$ [as shown in Fig. 1(a)] or $k_i=k_{av}=36$ [for clarity Fig. 1(b) features a network with $k_i=10$ just to exemplify the extension toward networks with larger degrees]. We thus contrast results obtained on scale-free networks with those obtained on regular networks having the same average degree $k_{av}=4$, and also with regular networks incorporating oscillators with k_i roughly the same as is the average maximal degree (≈ 35) that is encountered by the above-described scale-free networks.

For each set of the two main parameters ε and D the temporal output of each unit i is recorded for $T=1000$ periods of the pacemaker (after 100 periods were discarded as transients). Notably, to eliminate the effect of intrawell motion only every 20th point of the temporal series of x_i resulting from numerical integration is recorded. Finally, the correlation of each series with the frequency of the pacemaker $\omega=2\pi/t_p$ is computed via the Fourier coefficients $Q_i = \sqrt{R_i^2 + W_i^2}$ according to [36]

$$R_i = \frac{2}{t_p T} \int_0^{t_p T} x_i \sin(\omega t) dt, \quad (2)$$

$$W_i = \frac{2}{t_p T} \int_0^{t_p T} x_i \cos(\omega t) dt. \quad (3)$$

Since the Fourier coefficients are proportional to the square of the spectral power amplification, we presently use Q_i as the measure for stochastic resonance. To evaluate the response of the whole network by different D , the average of Q_i over all oscillators, defined as $S=N^{-1}\sum Q_i$, will be used.

III. RESULTS

In what follows, we will systematically analyze effects of different ε and D on the noise-induced temporal dynamics of the scale-free network of bistable oscillators. Throughout this section the results for the two considered pacemaker placements within the network, $r=i(k_{min})$ and $r=i(k_{max})$, will be shown and commented on in a parallel fashion for the purpose of better comparison. After presenting results for the scale-free networks, we will show results obtained on regular networks in order to strengthen the importance of degree inhomogeneity for the phenomenon of stochastic resonance.

We start by examining the color-contour plots of Q_i in dependence on D and i for different ε and r that are presented in Fig. 2. For the purpose of interpreting the presented results, it is important to distinguish between the so-called globally optimal response and the locally optimal response of the network. The globally optimal response of the network is obtained when S exhibits the overall maximum value brought about among all considered D , which directly implies that then all oscillators must contribute optimally to S via their individual Q_i . Instances of the globally optimal response are marked with yellow lines in all panels of Fig. 2, thus confirming the possibility of stochastic resonance within the presently considered model, irrespective of ε and r . Conversely to the globally optimal response, the locally optimal response is characterized by a (typically small) local maximum of S that is obtained by a smaller D , whereby only some small fraction of oscillators contribute significantly to S via their Q_i , whilst others remain largely uncorrelated with the pacemaker. The locally optimal response can be observed only for $r=i(k_{max})$ and small ε (up to $\varepsilon=0.08$), as exemplified by the red rectangle in the uppermost right panel of Fig. 2. It can be inferred that then the resonant response of the system is constrained to only roughly 30–40 oscillators that are directly connected to the one hosting the pacemaker. Note that the most likely maximal degree for the presently

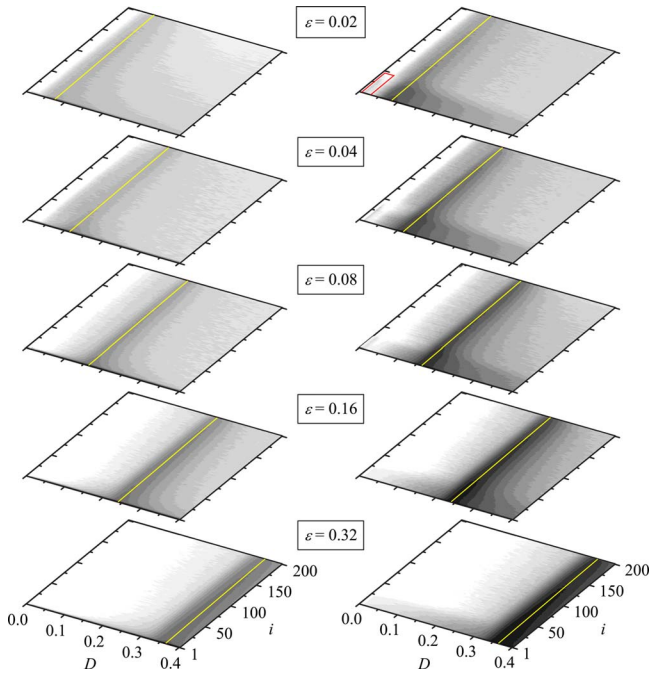


FIG. 2. (Color online) Color-coded Q_i in dependence on D and i for different ε . Left column features results for $r=i(k_{\min})$ and right for $r=i(k_{\max})$. Red rectangle in the uppermost right panel marks the area of the locally optimal response. Yellow lines denote the globally optimal response. The color profile in all panels is logarithmic, white depicting minimal and black maximal values of Q_i . Minimal Q_i in all panels is 0.001, whereas the maximal values from top to bottom for the left and right columns are 0.64,0.59,0.51,0.34,0.22 and 0.42,0.24,0.14,0.08,0.06, respectively. The consecutive numbering of vertices constituting the scale-free network is ordered in accordance with the area under each Q_i vs D curve. Since a larger area indicates a better-correlated response, the oscillator hosting the pacemaker always has $i=1$ whereas the oscillator that is most badly correlated with the pacemaker has $i=200$.

considered size of the scale-free network (≈ 35) corresponds rather accurately with the number of oscillators constituting the locally optimal response. Remnants of this feature uniquely present in case $r=i(k_{\max})$ (right column of Fig. 2) can be observed also for higher D and larger ε since there exists a clearly noticeable drop in magnitude of Q_i for all $i > 40$. If $r=i(k_{\min})$ (left column of Fig. 2), however, the locally optimal response is practically negligible, or at least cannot be appreciated within the graphical presentation of Fig. 2. It is also worth noting that, as a common feature independent of r , larger ε facilitates the outreach of the pacemaker across all coupled units (note that the darker shades extend increasingly toward higher i as ε increases), and, moreover, increasingly higher D are required for the optimal response. Accordingly, increasing ε also decrease the peak values of Q_i , as can be inferred from the caption of Fig. 2.

In order to study the above-outlined details of results presented in Fig. 2 more precisely, we continue by examining characteristic cross sections of the color maps as well as S in dependence on D . We start by showing Q_i in dependence on D for different ε for the oscillator $i=r$ that is under the direct influence of the pacemaker. The upper panel of Fig. 3 fea-

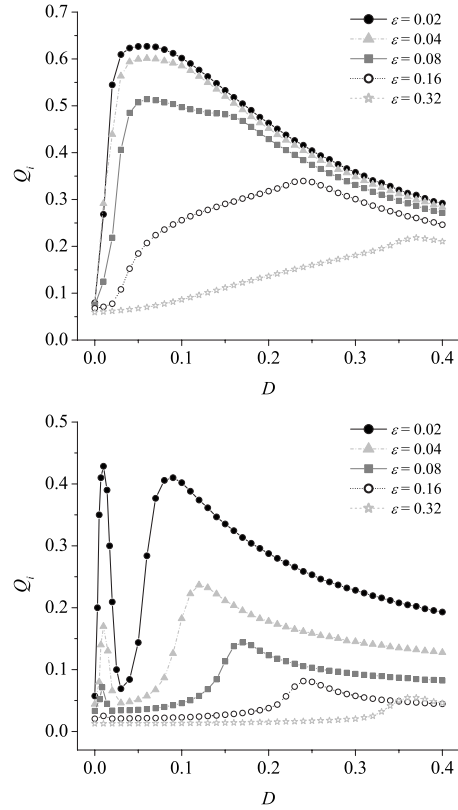


FIG. 3. Q_i in dependence on D for the oscillator $i=r$ that hosts the pacemaker. Upper panel shows results for $r=i(k_{\min})$ and lower for $r=i(k_{\max})$ by various ε . Lines are solely guides to the eye.

tures results for $r=i(k_{\min})$ and the lower panel for $r=i(k_{\max})$. The double-resonant response can be observed at a glance in the lower panel, where the locally optimal response (comprising foremost the pacemaker-driven oscillator and its direct neighbors) generates the first clear maximum of Q_i for substantially lower D than that subsequently giving the second maximum of Q_i representing the globally optimal response as the noise intensity increases further. However, as the coupling strength increases the locally optimal response vanishes quickly, which suggests that larger ε increasingly blur the details of the scale-free network structure, making the whole system essentially behave more and more like a single oscillator. Put differently, only small enough ε allow for the most efficient exploitation of the scale-free structure and its power-law degree distribution. Similar observations were made already when the stochastic resonance on small-world networks was studied [17], where it has also been argued that, in order for the complex network topology to take effect, an intermediate coupling strength is required. Importantly, however, even if the pacemaker is introduced to an oscillator with the lowest degree, the transition from the locally to the globally optimal response can be inferred, only that it does not result in a double resonance as D increases, but manifests as a smooth, yet rather abrupt, transition of the overall peak value of Q_i toward higher D as ε increases (see in particular the curves for $\varepsilon=0.08$ and 0.16 in the upper panel of Fig. 3). It is also noteworthy that the overall maximal peak of Q_i at smaller ε is larger if $r=i(k_{\min})$ than if $r=i(k_{\max})$. This must be attributed to the fact that when r

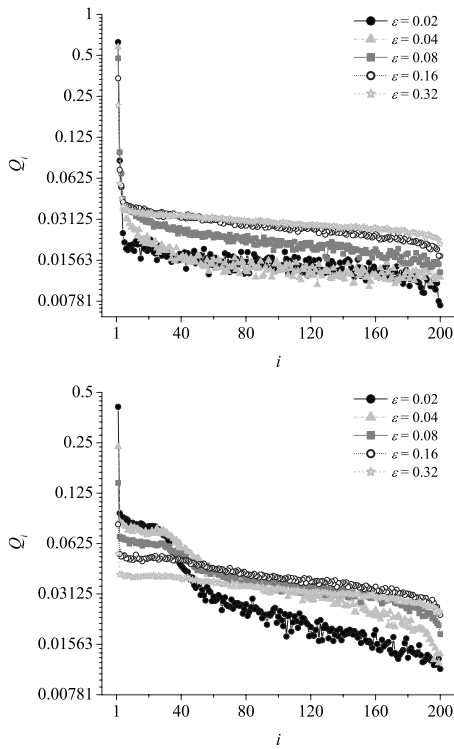


FIG. 4. Q_i in dependence on i for the value of D warranting the globally optimal response of the network (see yellow line in Fig. 2) for a given ϵ . Upper panel shows results for $r=i(k_{\min})$ and lower for $r=i(k_{\max})$. Note that in both cases the vertical axis has logarithmic scale. Lines are solely guides to the eye.

$=i(k_{\min})$ the local neighborhood of the paced unit (only two directly linked oscillators) is much smaller, and thus the influence of the pacemaker is much more weakly dissipated than if $r=i(k_{\max})$. Note that in the latter case up to 70 oscillators may be directly linked to the paced hub, essentially distorting its response.

Next, we show Q_i in dependence on i for the value of D warranting the globally optimal response of the network by a given ϵ . The lowermost right panel of Fig. 2 features a yellow line at the appropriate $D=0.37$ to exemplify an instance of the globally optimal response. We note, however, that $D=0.37$ is characteristic only for $\epsilon=0.32$, whereas the globally optimal response shifts toward smaller D as ϵ decreases, as can be inferred from Fig. 2, but also from results presented in Fig. 3 as well as Fig. 5 below. As in Fig. 3, the upper panel of Fig. 4 features results for $r=i(k_{\min})$ and the lower for $r=i(k_{\max})$. For $r=i(k_{\min})$ the trend is fairly straightforward in that larger ϵ improves the outreach of the pacemaker across all coupled units at the expense of an ever worse response of the directly paced oscillator. This obeys the simple and well-established reasoning that larger ϵ better dissipate the influence of the pacemaker, while simultaneously the directly perturbed oscillator is more strongly influenced by its surroundings and therefore its response becomes increasingly distorted. On the other hand, the curves depicted in the lower panel of Fig. 4 are somewhat more difficult to interpret. First, it is interesting to notice that the top 30–40 oscillators that are optimally correlated with the pacemaker behave exactly opposite in dependence on ϵ from the rest of the

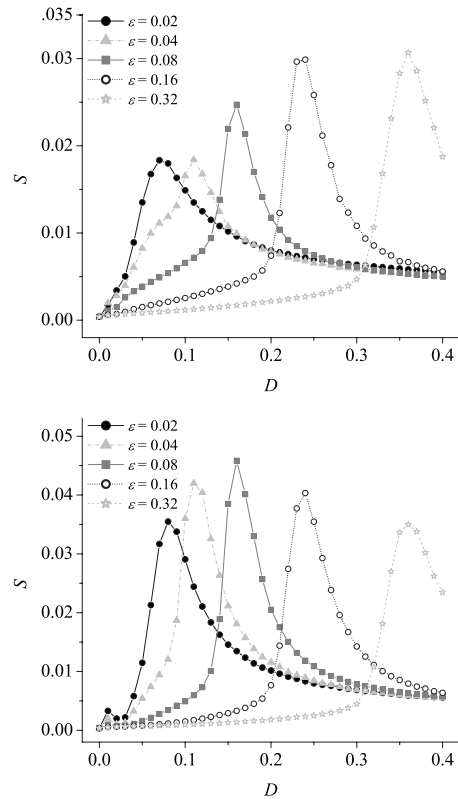


FIG. 5. S in dependence on D . Upper panel shows results for $r=i(k_{\min})$ and lower for $r=i(k_{\max})$ by various ϵ . Lines are solely guides to the eye.

coupled units. Namely, as ϵ increases, their correlation with the pacemaker decreases whilst the majority gradually improve their response. This, however, is again in accordance with the observation that only small ϵ allow for the occurrence of the locally optimal response, whereas larger ϵ increasingly blur the details of the scale-free network structure, making the whole system essentially behave more and more like a single unit. Remarkably, during this transition, as ϵ increases, there exists an optimal coupling strength which, on one hand, still allows the exploitation of the scale-free structure, but, on the other, also allows an efficient dissipation of the pacemaker-imposed rhythm to units that are not direct neighbors of the main hub. Note that the curve for $\epsilon=0.08$ seems to warrant the best compromise between the locally optimal response (response of those units that are directly connected to the pacemaker-driven unit) and the global response of the whole network. Such rather subtle features are absent in case $r=i(k_{\min})$.

To support the existence of an optimal ϵ in the case $r=i(k_{\max})$ we consider S (as defined in Sec. II) in dependence on D for different ϵ . Figure 5 shows the results for $r=i(k_{\min})$ (upper panel) and $r=i(k_{\max})$ (lower panel). Indeed, if the oscillator with the lowest connectivity is paced the optimal response (maximal S) of the network shifts toward higher D and improves as ϵ increases. In particular, the inability of the paced unit $r=i(k_{\min})$ to directly affect a large number of distant oscillators is at larger ϵ compensated by a faster transmission of the pacemaker rhythm across the whole array, which ultimately results in larger maximal S for

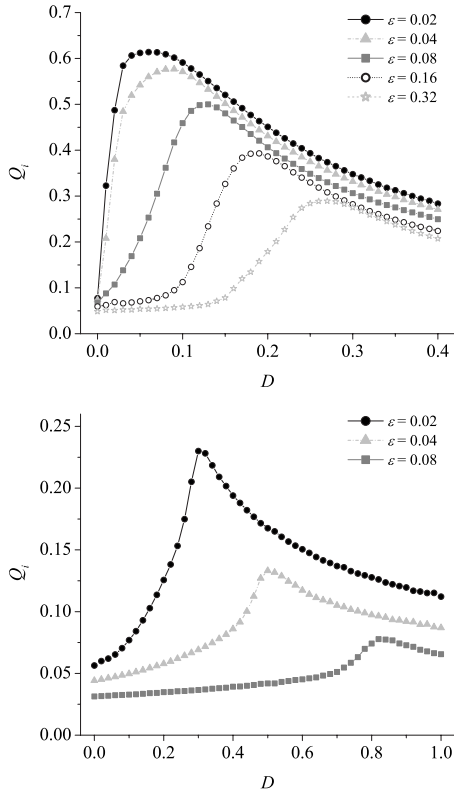


FIG. 6. Q_i in dependence on D for the oscillator $i=r$ that hosts the pacemaker on regular ring networks exemplified in Fig. 1. Upper panel shows results for $k_i=k_{av}=4$ and lower for $k_i=k_{av}=36$ for various ε . Lines are solely guides to the eye.

optimal D . On the other hand, if the paced oscillator is the main hub of the network then $\varepsilon=0.08$ warrants the largest overall peak of S , thus confirming the existence of an optimal ε for the global response of a weakly paced scale-free network under the influence of noise. Notably, this effect resembles the phenomenon of array-enhanced stochastic resonance reported in [5]. It is also noteworthy that the locally optimal response in the case $r=i(k_{max})$ is barely visible via S . For the locally optimal response, lower ε are preferred, and, indeed, very small humps around $D=0.01$ can be inferred only up to $\varepsilon=0.08$. Nevertheless, the locally optimal response should mainly be viewed as a prelude to the global one, since the latter uniquely quantifies the pacemaker’s efficiency and the ability of a network structure to support it.

Lastly, we consider regular networks, as depicted in Fig. 1, in order to strengthen the importance of the inhomogeneous scale-free network structure. As noted in Sec. II, we contrast the above results with those obtained on regular networks having $k_i=k_{av}=4$, and also with those that incorporate oscillators with $k_i=k_{av}=36$, which is roughly the same as the average maximal degree (≈ 35) encountered in the scale-free networks used above.

Figure 6 shows Q_i in dependence on D for different ε for the oscillator $i=r$ that is under the direct influence of the pacemaker. Note that, since regular rings with periodic boundary conditions are used, the particular placement of the pacemaker within such networks is irrelevant (we used $i=r=1$). The upper panel of Fig. 6 features results for $k_i=k_{av}$

$=4$ and the lower panel for $k_i=k_{av}=36$. First, it is important to note the similarities between results presented above in Fig. 3 and results presented in Fig. 6. In the upper panels of both figures the optimal D increases with increasing ε , and accordingly the peak values of Q_i decrease. Similar common features can be inferred also by comparing the lower panels of Figs. 3 and 6, only that the locally optimal response is absent in the case of the regular network. This indicates that regular networks fail to produce the locally optimal response as was reported above for the scale-free network if $r=i(k_{max})$. We argue that an increase in the degree of regular networks has a similar impact as if ε were to increase because the additional connections among distant oscillators simply facilitate the outreach of the pacemaker across the whole ensemble. Therefore, a qualitative change in the properties of the stochastic resonance phenomenon, possibility similar as reported above for scale-free networks if $r=i(k_{max})$, can thereby not be induced. We note that on scale-free networks vertices directly connected to $r=i(k_{max})$ will typically have a much smaller degree than the main hub itself due to the strongly inhomogeneous degree distribution. This fact implies that, once the pacemaker emitted signal reaches the first neighbors of $r=i(k_{max})$, the effective transmission of the pacemaker rhythm toward more distant neighbors deteriorates substantially, which leads to the existence of the locally optimal response by small enough ε . By regular networks with large k_i , on the other hand, all first neighbors of the paced oscillators have the same high degree, and thus very effectively dissipate the dynamics further to their neighbors (and this reasoning continues also for the third, fourth, etc., neighbors), and hence the locally optimal response is negligible even if ε is small.

As the last result corroborating our reasoning specific to results presented in Fig. 6, we show in Fig. 7 the quantity S dependent on D by different ε . Notably, many similar features as reported above for scale-free networks (see Fig. 5) can be observed. In particular, irrespective of the underlying network structure larger ε increase the optimal D as well as the maximal S , as shown in the upper panels of Figs. 5 and 7. The two lower panels of Figs. 5 and 7 also share the common feature that the optimal D increases with increasing ε . However, unlike in Fig. 5, in the lower panel of Fig. 7 there does not exist an optimal ε by which S would be maximal, which we attribute to the lack of features reported above for scale-free networks when the pacemaker is introduced to the main hub. Also, it can be noted that by $k_i=k_{av}=36$ increasing ε fail to enhance S markedly as the effective transmission of pacemaker activity is already warranted by the strong interconnectedness of individual oscillators. Our findings thus indicate that the strongly inhomogeneous scale-free network structure offers possibilities for optimizing the noise-supported transmission of localized rhythmic activity that cannot in this form be exploited effectively on regular networks, thus offering interesting insights into the stochastic resonance phenomenon on locally paced complex networks.

IV. SUMMARY

We study the phenomenon of stochastic resonance on locally paced scale-free networks of bistable oscillators. We

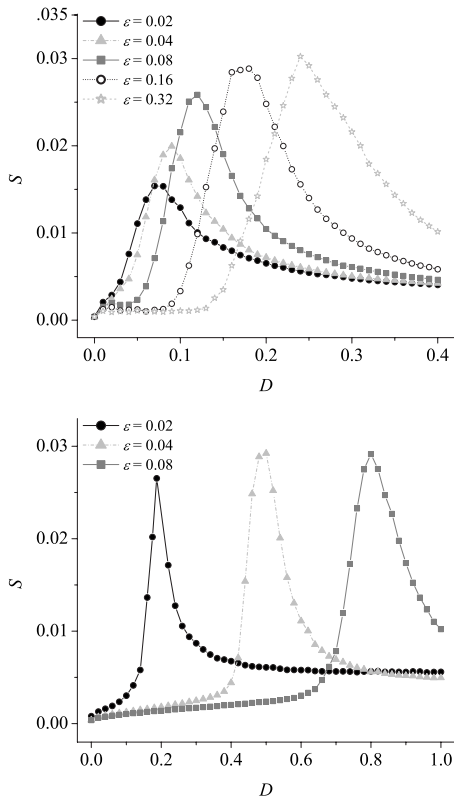


FIG. 7. S in dependence on D . Results were obtained on regular ring networks. Upper panel shows results for $k_i=k_{av}=4$ and lower for $k_i=k_{av}=36$ for various ε . Lines are solely guides to the eye.

find that an appropriate intensity of noise is able to induce an optimal correlation between the response of the network and the locally imposed rhythm, as depicted in Fig. 5. However, details of the reported stochastic resonance phenomenon depend extensively on the coupling strength and the degree of the oscillator hosting the pacemaker. If the pacemaker is introduced to the main hub of the network we can observe a so-called locally optimal response at small enough ε , by which only those oscillators that are directly linked with the one hosting the pacemaker adjust their response in accordance with the periodic forcing. This results in a double-resonant response as D increases, and, even more importantly, postulates the existence of an optimal coupling strength ε giving the best noise-induced global response of a scale-free network to a localized source of rhythmic activity. It is noteworthy that a double-resonant response in the spec-

tral power amplification has been reported already previously in an uncoupled bistable system [37] and in the Ising model on a scale-free network [27]. In the former case the double-resonant response was attributed to subharmonic clocking, whereas in the latter, which subsequently led also to the discovery of a dynamical phase transition in the Ising model on a scale-free network [38], the phenomenon was found to be dependent on the selection of the order parameter, i.e., the double-resonant response was observed only if the time-dependent magnetization was used for the evaluation of the spectral power amplification. At present, the optimal ε allows for an efficient dissipation of the pacemaker-imposed rhythm to units that are not direct neighbors of the main hub, but also is still low enough to allow the exploitation of the scale-free structure, thereby sustaining also some portion of the locally optimal response. On the other hand, if the pacemaker is introduced to one of the oscillators with the lowest connectivity, the small number of direct neighbors precludes the observation of an isolated locally optimal response, and consequently there does not exist an optimal ε warranting the best response in such a case. In fact, as the coupling increases the network then simply acts more and more as a single unit, and consequently the pacemaker becomes increasingly more successful in imposing its rhythm on the majority. Nevertheless, due to a much smaller neighborhood of directly linked oscillators, the pacemaker-driven unit can exhibit a better correlation with the pacemaker than if the main hub is paced, as depicted in Fig. 3, although this fact is mostly of secondary importance as it does not reflect the overall response of the whole network. We additionally strengthen the importance of degree inhomogeneity by studying the phenomenon of stochastic resonance on regular networks. While on the latter several features of the stochastic resonance phenomenon are similar to those reported for scale-free networks, regular networks do not seem to warrant the observation of a locally optimal response irrespective of the degree of the oscillators. Since pacemakers are integral parts of several different organs and cell types, we hope our study will find applicability in real-life-motivated problems and foster the understanding of biological as well as artificial processes that rely on an effective pacemaker for their proper functioning.

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[1] C. Nicolis and G. Nicolis, *Tellus* **33**, 225 (1981); R. Benzi, A. Sutera, and A. Vulpiani, *J. Phys. A* **14**, L453 (1981); L. Martínez, T. Pérez, C. R. Mirasso, and E. Manjarrez, *J. Neurophysiol.* **97**, 4007 (2007); E. Manjarrez, I. Mendez, L. Martínez, A. Flores, and C. R. Mirasso, *Neurosci. Lett.* **415**, 231 (2007); M. Perc, M. Gosak, and S. Kralj, *Soft Matter* **4**, 1861 (2008). Examples of reviews are P. Jung, *Phys. Rep.* **234**, 175 (1993); Special issue of *J. Stat. Phys.* **70**, 1 (1993), edited by

F. Moss, A. Bulsara, and M. F. Shlesinger; L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998); B. Lindner, J. García-Ojalvo, A. Neiman, and L. Schimansky-Geier, *Phys. Rep.* **392**, 321 (2004).

[2] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984); P. Hänggi and R. Bartussek, in *Nonlinear Physics of Complex Systems*, edited by J. Parisi, S. C. Müller, and W. Zimmermann (Springer, New York, 1999).

- [3] D. Sigeiti and W. Horsthemke, *J. Stat. Phys.* **54**, 1217 (1989); Gang Hu, T. Ditzinger, C. Z. Ning, and H. Haken, *Phys. Rev. Lett.* **71**, 807 (1993); W. J. Rappel and S. H. Strogatz, *Phys. Rev. E* **50**, 3249 (1994); A. S. Pikovsky and J. Kurths, *Phys. Rev. Lett.* **78**, 775 (1997); A. Longtin, *Phys. Rev. E* **55**, 868 (1997).
- [4] H. S. Wio, *Phys. Rev. E* **54**, R3075 (1996).
- [5] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, *Phys. Rev. Lett.* **75**, 3 (1995).
- [6] M. Löcher, G. A. Johnson, and E. R. Hunt, *Phys. Rev. Lett.* **77**, 4698 (1996); J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, *Phys. Rev. E* **53**, 2081 (1996); D. R. Chialvo, A. Longtin, and J. Muller-Gerking, *ibid.* **55**, 1798 (1997); N. Sungar, J. P. Sharpe, and S. Weber, *ibid.* **62**, 1413 (2000).
- [7] S. K. Han, T. G. Yim, D. E. Postnov, and O. V. Sosnovtseva, *Phys. Rev. Lett.* **83**, 1771 (1999); A. Neiman, L. Schimansky-Geier, A. Cornell-Bell, and F. Moss, *ibid.* **83**, 4896 (1999); C. S. Zhou, J. Kurths, and B. Hu, *ibid.* **87**, 098101 (2001).
- [8] P. Jung and G. Mayer-Kress, *Phys. Rev. Lett.* **74**, 2130 (1995); P. Jung, *ibid.* **78**, 1723 (1997); J. García-Ojalvo and L. Schimansky-Geier, *Europhys. Lett.* **47**, 298 (1999); J. Wang, *Chem. Phys. Lett.* **339**, 357 (2001); H. Busch and F. Kaiser, *Phys. Rev. E* **67**, 041105 (2003); E. Ullner, A. A. Zaikin, J. García-Ojalvo, and J. Kurths, *Phys. Rev. Lett.* **91**, 180601 (2003); O. Carrillo, M. A. Santos, J. García-Ojalvo, and J. M. Sancho, *Europhys. Lett.* **65**, 452 (2004); M. R. Roussel and J. Wang, *J. Chem. Phys.* **120**, 8079 (2004); C. S. Zhou and J. Kurths, *New J. Phys.* **7**, 18 (2005); E. Glatt, H. Busch, F. Kaiser, and A. Zaikin, *Phys. Rev. E* **73**, 026216 (2006); Q. Y. Wang, Q. S. Lu, and G. R. Chen, *Europhys. Lett.* **77**, 10004 (2007); O. Nekhamkina and M. Sheintuch, *Phys. Rev. E* **75**, 056210 (2007); M. Perc and A. Szolnoki, *New J. Phys.* **9**, 267 (2007); X. Sun, M. Perc, Q. Lu, and J. Kurths, *Chaos* **18**, 023102 (2008).
- [9] F. Sagués, J. M. Sancho, and J. García-Ojalvo, *Rev. Mod. Phys.* **79**, 829 (2007).
- [10] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002); S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, *Phys. Rep.* **424**, 175 (2006).
- [11] D. J. Watts and S. H. Strogatz, *Nature (London)* **393**, 440 (1998).
- [12] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).
- [13] S. Wasserman and K. Faust, *Social Network Analysis* (Cambridge University Press, Cambridge, U.K., 1994).
- [14] A. F. J. Van Raan, *Nature (London)* **347**, 626 (1990); P. O. Seglen, *J. Am. Soc. Inf. Sci.* **43**, 628 (1992); S. Redner, *Eur. Phys. J. B* **4**, 131 (1998); M. E. J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **101**, 5200 (2004).
- [15] K. McCann, A. Hastings, and G. R. Huxel, *Nature (London)* **395**, 794 (1998).
- [16] B. A. Huberman and L. A. Adamic, *Nature (London)* **401**, 131 (1999); L. A. Adamic, B. A. Huberman, A.-L. Barabási, R. Albert, H. Jeong, and G. Bianconi, *Science* **287**, 2115 (2000).
- [17] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, *Phys. Rev. Lett.* **84**, 2758 (2000); O. Shefi, I. Golding, R. Segev, E. Ben-Jacob, and A. Ayali, *Phys. Rev. E* **66**, 021905 (2002); A. Roxin, H. Riecke, and S. A. Solla, *Phys. Rev. Lett.* **92**, 198101 (2004); V. M. Eguíluz, D. R. Chialvo, G. A. Cecchi, M. Baliki, and A. V. Apkarian, *ibid.* **94**, 018102 (2005); V. Volman, I. Baruchi, and E. Ben-Jacob, *Phys. Biol.* **2**, 98 (2005); P. K. Swain and A. Longtin, in *Stochastic Dynamics of Neural and Genetic Networks*, edited by A. Longtin and P. K. Swain, special focus issue of *Chaos*, **16**, 026101 (2006); N. Masuda, M. Okada, and K. Aihara, *Neural Comput.* **19**, 1854 (2007).
- [18] Z. Gao, B. Hu, and G. Hu, *Phys. Rev. E* **65**, 016209 (2001); H. Hong, B. J. Kim, and M. Y. Choi, *ibid.* **66**, 011107 (2002); M. Perc and M. Gosak, *New J. Phys.* **10**, 053008 (2008).
- [19] O. Kwon, H.-T. Moon, *Phys. Lett. A* **298**, 319 (2002); O. Kwon, H.-H. Jo, and H.-T. Moon, *Phys. Rev. E* **72**, 066121 (2005).
- [20] D. He, G. Hu, M. Zhan, W. Ren, and Z. Gao, *Phys. Rev. E* **65**, 055204(R) (2002); X. Wang, Y. Lu, M. Jiang, and Q. Ouyang, *ibid.* **69**, 056223 (2004); M. Perc, *New J. Phys.* **7**, 252 (2005).
- [21] Y. Gong, B. Xu, Q. Xu, C. Yang, T. Ren, Z. Hou, and H. Xin, *Phys. Rev. E* **73**, 046137 (2006); D. Q. Wie and X. S. Luo, *Europhys. Lett.* **78**, 68004 (2007).
- [22] T.-H. Tsao and R. Butera, *Neural Comput.* **18**, 2029 (2006).
- [23] M. Kuperman and D. Zanette, *Eur. Phys. J. B* **26**, 387 (2002).
- [24] J. A. Acebrón, S. Lozano, and A. Arenas, *Phys. Rev. Lett.* **99**, 128701 (2007).
- [25] C. Zhou and J. Kurths, *Phys. Rev. Lett.* **96**, 164102 (2006); A. Arenas, A. Diaz-Guilera, and C. J. Perez-Vicente, *Physica D* **224**, 27 (2006); C. Zhou, A. E. Motter, and J. Kurths, *Phys. Rev. Lett.* **96**, 034101 (2006); J. Gomez-Gardeñes, Y. Moreno, and A. Arenas, *ibid.* **98**, 034101 (2007).
- [26] A. Krawiecki, *Physica A* **333**, 505 (2004).
- [27] A. Krawiecki, *Int. J. Mod. Phys. B* **18**, 1759 (2004).
- [28] A. M. Katz, *Physiology of the Heart* (Kluwer, Philadelphia, 2000).
- [29] R. E. Haddock and C. E. Hill, *J. Physiol. (London)* **566**, 645 (2005).
- [30] R. Dumollard, J. Carroll, G. Dupont, and C. Sardet, *J. Cell. Sci.* **115**, 3557 (2002).
- [31] Y. Nagai, H. González, A. Shrier, and L. Glass, *Phys. Rev. Lett.* **84**, 4248 (2000); S. Alonso, I. Sendiña-Nadal, V. Pérez-Muñuzuri, J. M. Sancho, and F. Sagués, *ibid.* **87**, 078302 (2001); M. Gutman, I. Aviram, and A. Rabinovitch, *Phys. Rev. E* **70**, 037202 (2004); L. B. Smolka, B. Marts, and A. L. Lin, *ibid.* **72**, 056205 (2005); V. Jacquemet, *ibid.* **74**, 011908 (2006); M. Perc and M. Marhl, *Phys. Lett. A* **353**, 372 (2006); T. R. Chigwada, P. Parmananda, and K. Showalter, *Phys. Rev. Lett.* **96**, 244101 (2006); M. Perc, *Phys. Rev. E* **76**, 066203 (2007).
- [32] H. Kori and A. S. Mikhailov, *Phys. Rev. Lett.* **93**, 254101 (2004); F. Radicchi and H. Meyer-Ortmanns, *Phys. Rev. E* **73**, 036218 (2006); A. J. Steele, M. Tinsley, and K. Showalter, *Chaos* **16**, 015110 (2006).
- [33] R. Zhang, L. Yang, A. M. Zhabotinsky, and I. R. Epstein, *Phys. Rev. E* **76**, 016201 (2007).
- [34] A. Pikovsky, A. Zaikin, and M. A. de la Casa, *Phys. Rev. Lett.* **88**, 050601 (2002).
- [35] C. J. Tessone, C. R. Mirasso, R. Toral, and J. D. Gunton, *Phys. Rev. Lett.* **97**, 194101 (2006).
- [36] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical recipes in C* (Cambridge University Press, Cambridge, U.K., 1995).
- [37] P. Jung and P. Hänggi, *Phys. Rev. A* **44**, 8032 (1991).
- [38] A. Krawiecki, *Int. J. Mod. Phys. B* **19**, 4769 (2005).