Nonlinear Dynamics
https://doi.org/10.1007/s11071-025-11911-9

RESEARCH

Predicting bifurcation points in continuous chaotic neuron models

Shivakumar Rajagopal[1,2] · Fahimeh Nazarimehr[3] · Karthikeyan Rajagopal[2,4] · Sajad Jafari[3,5] · Matjaž Perc[6,7,8,9]

Received: 1 October 2025 / Revised: 1 November 2025 / Accepted: 8 November 2025

Abstract

We study how to predict bifurcation points in continuous dynamical systems, with a particular focus on complex behaviors relevant to neuroscience. Standard early warning signals such as autocorrelation, variance, skewness, and kurtosis work well for simple bifurcations but often fail for transitions involving limit cycles, period-doubling, or chaos. Building on our earlier work with discrete maps, we adapt the period-based decomposition method to continuous flows. Our approach first samples the dynamics with a Poincaré section, then estimates the dominant period and decomposes the signal into sub-series. Early warning indicators are computed for each sub-series and averaged to produce modified signals. Applying this method to the Hindmarsh-Rose neuron model, we show that the modified indicators reveal clear and consistent trends before period-doubling and chaotic transitions, greatly improving on standard approaches. We also find that focusing on the slowest system variable is essential for obtaining reliable signals. Our results indicate that this framework provides a powerful tool for anticipating complex critical transitions in continuous dynamical systems and may help predict challenging dynamics in real-world settings.

Keywords Early warning signals · Bifurcation points · Bifurcations · Chaos · Hindmarsh-Rose neuron model · Continuous dynamical systems · Poincaré map · Critical transitions

1 Introduction

Understanding and predicting critical transitions, often referred to as tipping points, in complex dynamical systems is a fundamental challenge with broad implications across science and engineering [1, 2]. These transitions signify abrupt shifts in a system's behavior, where it moves from one stable regime to another, often leading to significant and sometimes irreversible consequences [3, 4]. Such phenomena are observed in diverse contexts, including climate systems [3], ecological networks [5], financial markets [6], and the dynamics of the human brain [7–9]. The ability to anticipate these tipping points before they occur is therefore of paramount importance for mitigating risks and managing system behavior effectively [10].

In the context of neuroscience, critical transitions are particularly relevant for understanding pathological states and the mechanisms underlying changes in neural activity [11, 12]. Neuronal dynamics are inherently continuous in time, typically modeled by systems of ordinary differential equations (ODEs) that generate continuous flows [13–15] and can show various dynamics like chaos [16]. A variety

✉ Matjaž Perc
matjaz.perc@gmail.com

1 Center for Cognitive Science, Trichy SRM Medical College Hospital and Research Center, Trichy, India

2 Center for Research, Easwari Engineering College, Chennai, India

3 Department of Biomedical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran

4 Center for Research, SRM TRP Engineering College, Trichy, India

5 Health Technology Research Institute, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran

6 Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška Cesta 160, 2000 Maribor, Slovenia

7 Community Healthcare Center Dr. Adolf Drolc Maribor, Ulica Talcev 9, 2000 Maribor, Slovenia

8 Department of Physics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea

9 University College, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, Republic of Korea

Published online: 05 December 2025

of neuron models have been investigated [17–19], including network models [20–22] and specialized types like coupled memristive neural networks [23–25]. A survey of the nonlinear dynamics viewpoint in neuroscience was presented in [26–28]. Investigating networks of such models has been a hot topic [29–31]. Cluster synchronization is an interesting phenomenon that has attracted much attention [32, 33]. The flow-based models, such as the Hindmarsh-Rose (HR) model [34], are capable of reproducing a rich repertoire of behaviors, including stable resting states, periodic spiking and bursting, and intricate chaotic firing patterns [14, 35, 36]. Various dynamics of neural models have been studied in [37–39]. Transitions between these different dynamical regimes can be viewed as tipping points, and predicting them is essential for gaining insights into phenomena like epileptic seizures, changes in cognitive states, or the onset of neurological disorders.

A widely used approach for anticipating tipping points relies on early warning signals (EWS) [40]. These are statistical metrics derived from time series data that are designed to detect the phenomenon of critical slowing down—the tendency for a system's recovery from perturbations to become increasingly slow as it approaches a bifurcation point [41]. Common EWS include the autocorrelation function at lag-1 (ACC), variance, skewness, and kurtosis [41]. These indicators have proven effective in predicting simple bifurcations, such as the transition from a stable fixed point to another stable state. However, their applicability is significantly limited when dealing with more complex dynamical transitions, particularly those involving limit cycles, period-doubling cascades, or crises leading to chaos [42]. In such scenarios, standard EWS often fail to exhibit the expected trends, rendering them ineffective for reliable prediction. In our previous work, we addressed this limitation for discrete dynamical systems (maps) by proposing a modification to standard EWS [42]. This involved estimating the underlying period of the system's oscillation from the time series and then decomposing the original time series into sub-time series based on this period. Standard EWS were then calculated for each sub-series and averaged. This period-based decomposition significantly improved the prediction performance for complex transitions like period-doubling routes to chaos in maps.

Here, we extend this successful methodology to continuous dynamical systems (flows), specifically focusing on chaotic neuron models with period-doubling route to chaos bifurcations. Directly applying the discrete map method to high-resolution continuous data is computationally prohibitive and analytically problematic. We overcome this by first sampling the continuous flow using a Poincaré section, effectively transforming the continuous dynamics into a discrete map. We then adapt the period-based decomposition approach to this Poincaré sampled data. The proposed method is tested on the Hindmarsh-Rose neuron model, demonstrating its effectiveness in predicting tipping points associated with complex bifurcations in continuous flows, thereby bridging the gap between EWS methodologies for discrete and continuous systems.

The remainder of this paper is structured as follows. In Sect. 2, we detail the methodology, beginning with an introduction to the HR neuron model and the challenges associated with applying standard early warning signals directly to continuous flow data. We then describe the proposed method, which adapts our previous period-based decomposition approach for discrete maps by incorporating Poincaré sampling to handle the continuous nature of the flows. Section 3 presents the results of applying the modified early warning signals (Autocorrelation, Variance, Kurtosis, and Skewness) to the HR model, demonstrating their improved performance in predicting various bifurcation points compared to standard indicators. Section 4 provides a detailed discussion, focusing on the impact of variable selection for Poincaré sampling and the underlying reasons for the improved performance of the proposed method. Finally, Sect. 5 offers concluding remarks, summarizing the key findings and implications of this work for predicting critical transitions in continuous dynamical systems.

2 Methodology: forecasting bifurcation points in models of chaotic flow-based neurons

Since neural activity dynamics are naturally continuous in time, they are frequently represented by ordinary differential equation (ODE) systems that produce continuous flows [37]. These flow-based neuron models are capable of reproducing a rich repertoire of complex behaviors observed in real neural systems, including stable resting states, periodic spiking and bursting, and intricate chaotic firing patterns [43]. Such complex dynamics are particularly relevant in understanding pathological states and the transitions between different regimes of neural activity [44], which can be viewed as tipping points in the system's behavior. Gaining understanding of the mechanisms behind occurrences such as epileptic seizures, changes in cognitive states, or the onset of neurological illnesses requires the ability to predict these critical transitions in continuous neural models.

Autocorrelation, variance, skewness, and kurtosis are examples of standard early warning signals (EWS) that have been shown to be useful in predicting important transitions linked to basic bifurcations (e.g., approaching a

stable fixed point) in both maps and flows [41, 42]. In our previous work [42], we addressed the limitations of standard EWS for predicting bifurcation points in discrete maps undergoing period-doubling cascades. We proposed a modification where the time series is first analyzed to estimate the underlying period, and then decomposed into sub-time series. Standard EWS were then calculated for each sub-series and averaged. The approach significantly improved prediction performance for complex transitions in maps. However, when this method is applied directly to time series produced by continuous systems moving through more complicated bifurcation situations, like the period-doubling route to chaos or crises, these indicators frequently fall short (which is discussed in this section). The dynamics in flow systems are continuous and frequently oscillatory, which makes it difficult to consistently identify the statistical indicators of significant slowing down connected to these intricate transitions.

Here, we particularly adapt this effective methodology for use with continuous dynamical systems (flows), concentrating on neuron models that display complex bifurcations and chaotic dynamics. Adapting the two-step approach to deal with the continuous-time nature of the data is still the main premise. In the following, the challenges of the previous method [42] are discussed. Then, the improved method that can deal with those challenges is proposed.

2.1 The studied model

In this study, we utilize the well-known HR neuron model to investigate various bifurcation points in continuous-time neuronal dynamics. The HR model is a system of three coupled ordinary differential equations that effectively captures the complex firing patterns observed in biological neurons. The model is defined as:

$$\begin{aligned} \dot{x} &= y + bx^2 - x^3 - z + I, \\ \dot{y} &= 1 - 5x^2 - y, \\ \dot{z} &= -rz + rs(x - x_{rest}), \end{aligned} \tag{1}$$

where the variables x, y, and z represent the membrane potential, a recovery variable (or spiking variable), and a bursting variable, respectively. The parameter I represents the external forcing current (equivalent to membrane input current in biological neurons), while b controls the transition between bursting and spiking behaviors and influences spike frequency. The parameter r governs the efficiency rate of slow ion channels, controlling spike frequency during spiking activity and affecting the number of spikes per burst during bursting behavior. Additionally, s controls the matching between variables, and x_{rest} sets the system's resting potential.

For our analysis, we fix the parameters at $b = 3, I = 3.25, s = 4$, and $x_{rest} = -1.6$. The parameter r is selected as the primary bifurcation parameter (see Fig. 1) and is systematically varied within the interval $[0, 0.05]$ to explore a range of dynamical behaviors, including transitions from simple spiking to complex bursting patterns and chaotic dynamics. The diagrams are plotted with run time of 6000 and the resolution of changing parameter is 0.00001. This parameter range encompasses several critical transitions, making it suitable for evaluating the performance of our proposed early warning indicators in predicting bifurcation points in continuous dynamical systems. The HR model's rich bifurcation structure, including period-doubling routes to chaos, makes it an ideal testbed for extending our previously developed methodology from discrete maps to continuous flows.

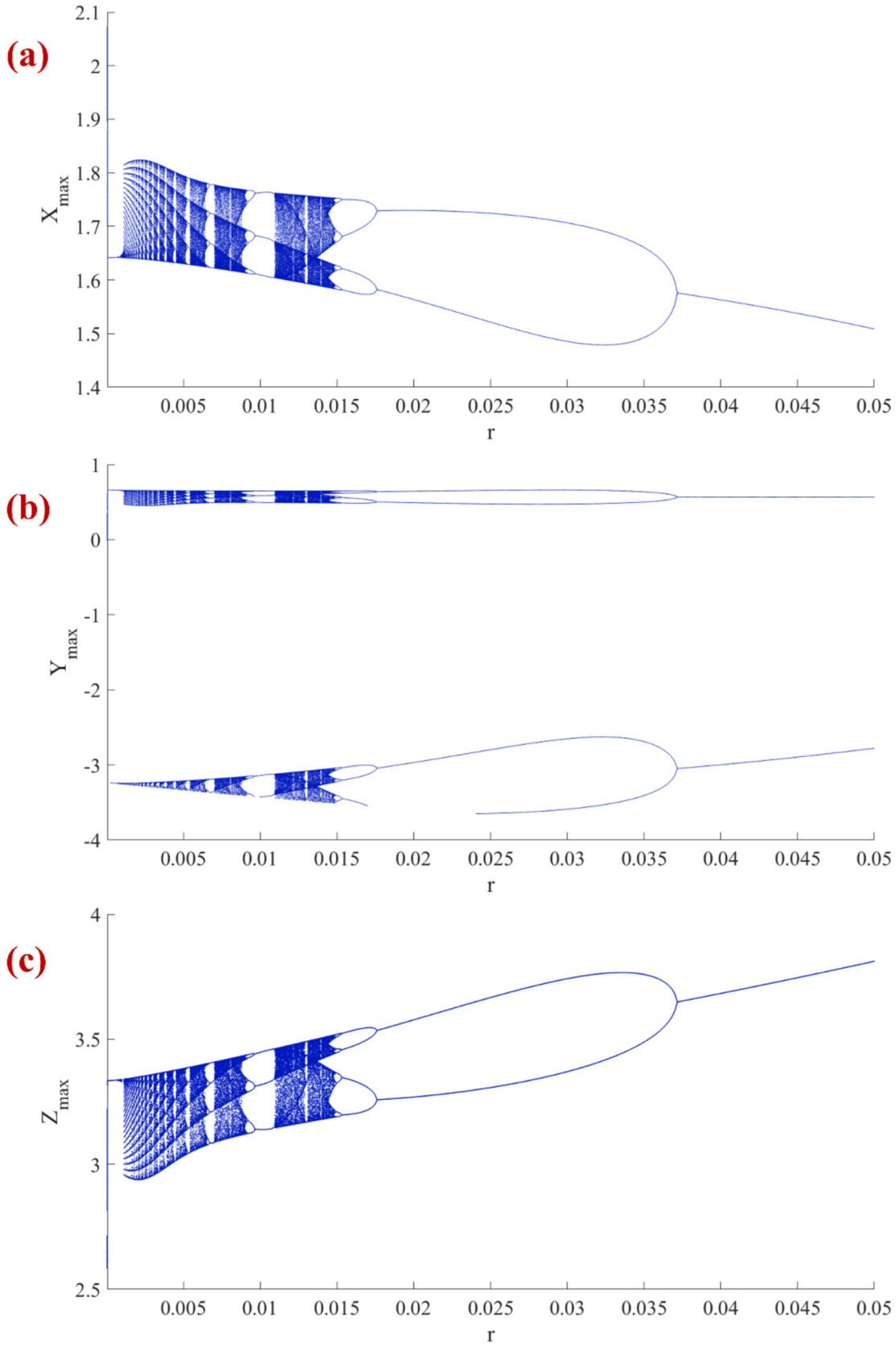
2.2 Challenges of the EWS in continuous-time dynamical systems

There are some challenges in applying the method of our previous work to the chaotic flows. To investigate these challenges, first we apply the standard EWS (autocorrelation, variance, skewness, and kurtosis) to the continuous-time neuronal dynamics of the HR model. Then we further discuss the challenges of applying the improved version of EWS to the chaotic flows.

2.2.1 Limitations of standard early warning signals in flow analysis

When numerically solving chaotic flows, achieving accurate results necessitates the use of small time step sizes. In this study, we employ MATLAB's ODE45 solver with a stringent error tolerance of 10^{-20} to ensure precise integration of the continuous dynamical system. This high accuracy requirement, while essential for capturing the true dynamics, introduces significant computational challenges for EWS analysis.

To illustrate these challenges, we focus on the autocorrelation function with lag-1 (ACC) as a representative standard EWS when applied to continuous-time neural systems. The primary computational difficulty arises from the nature of flow data: even for relatively short simulation run times, the resulting time series becomes extremely long due to the small integration time steps. For instance, with our chosen parameters and solver settings, a modest run time of 200 time units (in Fig. 2) generates an extensive dataset that poses substantial computational demands for EWS calculation. Figure 2 presents the analysis results for the HR system (1), where the autocorrelation with lag-1 is computed for each parameter value of r. The


(a)
2.1
2
1.9
1.8
1.7
1.6
1.5
1.4
X max
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
r
(b)
1
0
-1
-2
-3
-4
Y max
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
r
(c)
4
3.5
3
2.5
Z max
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
r

◀**Fig. 1** Bifurcation diagram of continuous time neuron model (1) by varying parameter r and forward continuation method; the first set of initial conditions are $(0.5,1,0)$. Bifurcation diagrams are plotted by maximum values of **a** x variable; **b** y variable; **c** z variable; the diagrams reveal a rich sequence of dynamical transitions—including period-1 spiking, period-doubling cascades, periodic windows, and chaotic regimes—providing a comprehensive map of tipping points used to evaluate early warning signals in this study

figure displays two key components: the computed ACC values from the z variable signal (shown in red) overlaid on the bifurcation diagram of maximum z values (shown in blue). This visualization highlights the fundamental limitations of standard EWS when applied to continuous flows.

The results reveal critical shortcomings of the standard ACC approach in the flow context. Most notably, the ACC fails to exhibit any meaningful trend as the system approaches bifurcation points or moves away from them. Unlike the clear patterns observed in discrete maps or simple continuous systems undergoing basic bifurcations, the ACC values for the flow system with more complex dynamics than equilibrium points remain approximately constant at unity across the entire parameter range. This persistent value of approximately 1 indicates that the standard autocorrelation method lacks sensitivity to the underlying dynamical changes occurring in the continuous system, rendering it ineffective for predicting bifurcation points in chaotic flow-based neuron models.

Similar limitations are observed for other standard EWS (variance, skewness, and kurtosis) when applied directly to flow data, motivating the need for the modified approach we propose in the following sections. These computational and conceptual challenges underscore the fundamental differences between analyzing discrete maps and continuous flows in the context of early warning signals for complex dynamical transitions.

2.2.2 Overcoming the limitation using the Poincaré map

Here to solve the issue, we should use a sampling method to remove the extra and unnecessary information from the time series. An excellent method to sample a dynamical system's time series is using the Poincaré section. Here we use the maximum values of the z variable as the Poincaré section and computed the ACC to see if there is any proper trend or not. Figure 3 shows the bifurcation plot using the maximum values of z variable and the computed ACC using the Poincaré signal. An important point can be seen in this figure that in the end of the interval of r parameter, the ACC have a proper trend when the dynamics are priod-1. It can be seen that by moving away from the bifurcation point in $r = 0.037$ the ACC decreases from 1. However, we cannot see any change when the system shows priod-2 dynamics in the interval $[0.018,0.037]$.

To address the challenges associated with applying early warning signals directly to high-resolution continuous flow data, we employ a sampling strategy that extracts essential

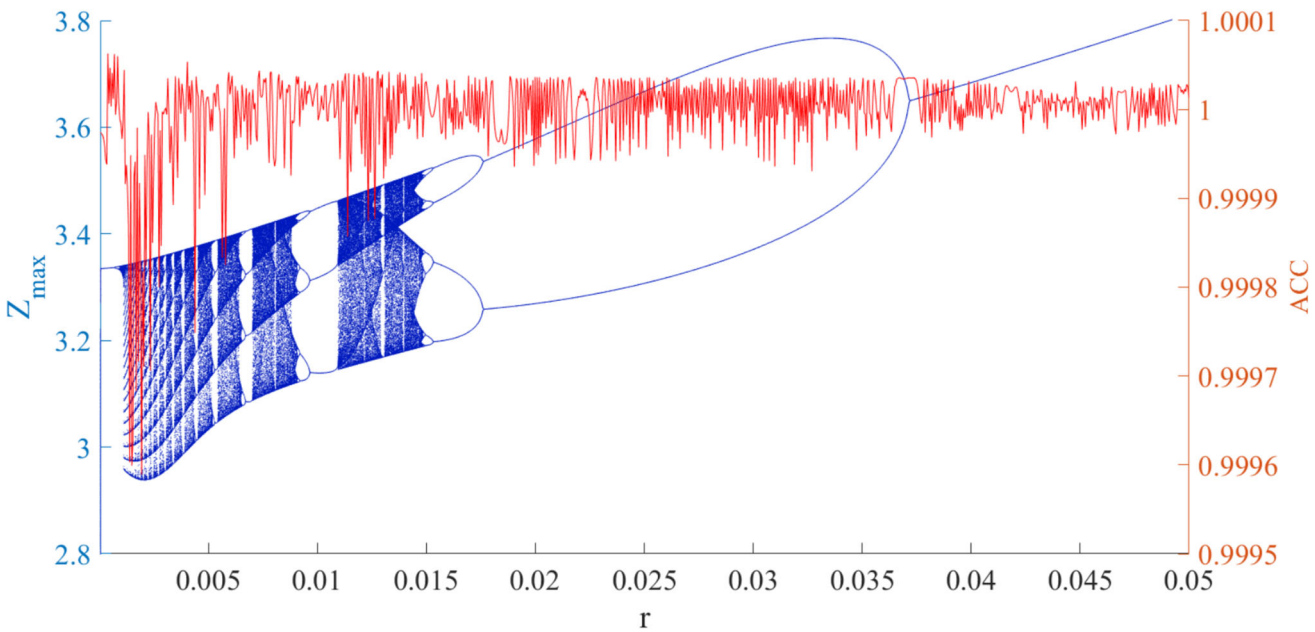


Fig. 2 Limitations of standard autocorrelation at lag-1 (ACC) when applied directly to the continuous time series of the HR model (variable z); the figure shows the computed ACC values (red) plotted against the bifurcation diagram of maximum z values (blue) for r ∈ [0, 0.05]. The ACC values (red) remain nearly constant at 1 across the entire parameter range, indicating that standard early warning signals fail to capture critical slowing down in continuous chaotic flows due to oversampling and lack of oscillation-aware decomposition

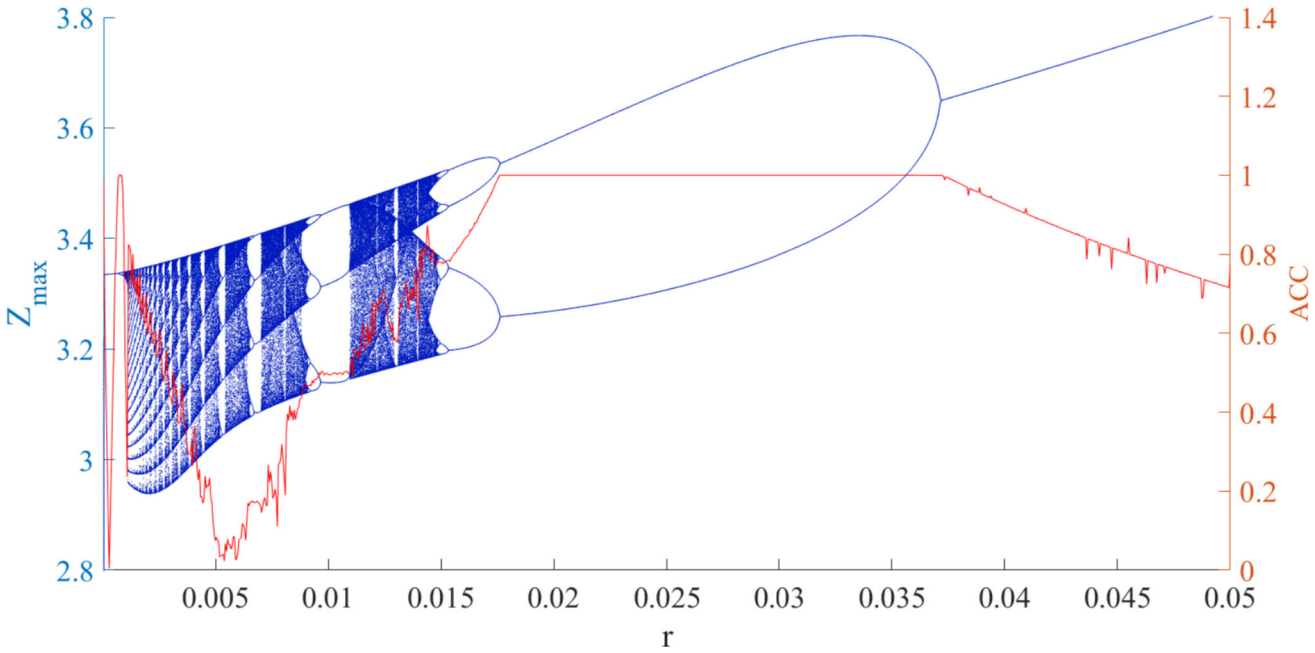


Fig. 3 Performance of standard autocorrelation at lag-1 (ACC) applied to the Poincaré sampled signal (local maxima of z variable) from the HR model. The computed ACC values (red) are plotted alongside the bifurcation diagram of maximum z values (blue). While a decreasing trend in ACC is visible near $r = 0.05$ for period-1 dynamics—consistent with critical slowing down—the indicator fails to show meaningful trends during period-2 or chaotic intervals, demonstrating that Poincaré sampling alone is insufficient for detecting complex bifurcations

dynamical information while eliminating redundant data points. The Poincaré section method provides an elegant approach for sampling continuous dynamical systems by capturing the system's state at regular intervals corresponding to specific geometric or temporal criteria [45].

In this study, we implement the Poincaré map by sampling the system state at the local maxima of the z variable trajectory. This approach effectively transforms the continuous flow into a discrete map that preserves the essential dynamical characteristics while significantly reducing the data volume. By focusing on the maximum values of z, we capture the recurrent behavior of the system's bursting dynamics, which is particularly relevant for the Hindmarsh-Rose neuron model.

Figure 3 presents the analysis results obtained using this Poincaré sampling approach. The figure displays the bifurcation diagram constructed from the maximum z values (shown in blue) alongside the autocorrelation coefficient with lag-1 computed from the Poincaré sampled signal (shown in red). This visualization reveals important insights into the effectiveness of the sampling method.

A critical observation from Fig. 3 is the improved behavior of the ACC when computed on the Poincaré sampled data compared to the raw continuous signal. Specifically, at the end of the parameter interval where the system exhibits period-1 dynamics (approaching $r = 0.05$), the ACC demonstrates a proper trend as the system moves away from the bifurcation point. In this regime, the ACC decreases from its maximum value of 1, indicating the expected behavior of critical slowing down indicators.

However, limitations persist even with the Poincaré sampling approach. During parameter intervals where the system exhibits period-2 dynamics, the ACC computed from the Poincaré map fails to show any meaningful trend that would indicate approaching bifurcation points. This observation highlights that while Poincaré sampling addresses the computational burden, it does not inherently resolve the fundamental limitation of standard early warning signals in detecting complex transitions such as period-doubling cascades.

These results underscore the need for further methodological modifications beyond simple sampling techniques. The Poincaré map approach, while useful for reducing data complexity, still requires the enhanced methodology we propose to effectively predict bifurcation points associated with complex bifurcation scenarios in continuous dynamical systems. This finding motivates our development of modified early warning indicators specifically adapted for flow systems, building upon the successful framework previously established for discrete maps.

2.3 The proposed method

Building upon the successful framework for predicting tipping points in discrete maps via period-based decomposition of time series [42], we adapt and extend this

methodology specifically for application to continuous dynamical systems (flows). The core principle remains: decompose the time series based on the estimated underlying periodicity to enhance the sensitivity of standard EWS to complex bifurcations.

However, applying this directly to raw, high-resolution flow data, as discussed in Sect. 2.1.1, is computationally prohibitive and analytically problematic. As demonstrated in Sect. 2.1.2, utilizing a Poincaré section (specifically, sampling the system state at the local maxima of the z variable in our example) effectively addresses the data volume issue and provides a discrete representation of the flow's dynamics. Nevertheless, the results in Sect. 2.1.2 also showed that applying standard EWS directly to this Poincaré sampled signal still fails to reliably predict complex transitions like period-doubling cascades.

Therefore, a modified approach is proposed that combines the benefits of Poincaré sampling with the enhanced sensitivity of our previously developed period-based decomposition method. The proposed method consists of the following steps, applied to the Poincaré sampled time series obtained from the continuous flow signal:

1. Computation of the Poincaré signal from the selected signal: The continuous time series obtained from the flow system (e.g., the variable $z(t)$ from the HR model) is processed to generate a discrete Poincaré section. This is achieved by sampling the continuous signal at specific, dynamically relevant points. In this study, we select the local maxima of the chosen variable (z) as the Poincaré section. This transformation effectively converts the continuous flow data into a discrete signal, significantly reducing data volume while preserving the essential dynamical information related to the system's recurrent behavior, making subsequent analysis computationally feasible and focusing on the underlying oscillatory structure.
2. Dynamic Type Estimation for Poincaré Series: The first step involves estimating the underlying dynamic type (i.e., the period of the oscillation) directly from the Poincaré sampled time series. We analyze the Poincaré time series to estimate its dominant period, denoted as T. For a flow sampled via Poincaré section, T represents the number of oscillation cycles (or the period of the attractor in the Poincaré map) between successive sampling points. If the continuous flow exhibits chaotic dynamics, the estimated T will typically reaches a predefined maximum threshold (Tr).
3. Decomposition of the Poincaré Time Series: Once the period T is estimated from the Poincaré sampled data, the second step involves decomposing this discrete Poincaré time series into T distinct sub-time series. Then, the original Poincaré time series z_P is divided into T sub-time series:

$$w_1 = \{z_P(1), z_P(1+T), z_P(1+2T), ...\}$$
$$w_2 = \{z_P(2), z_P(2+T), z_P(2+2T), ...\}$$
$$...$$
$$w_T = \{z_P(T), z_P(T+T), z_P(T+2T), ...\}$$

1. Calculation of Modified Early Warning Signals: Standard EWS (autocorrelation at lag-1, variance, skewness, kurtosis) are then calculated individually for each of the T sub-timseries $(w_1, ..., w_T)$ derived from the Poincaré map.
2. Averaging: Finally, the modified early warning indicator is obtained by averaging the values of the chosen EWS computed from the T sub-time series. For example, the modified autocorrelation is calculated as: $\frac{1}{T}\Sigma_{i=1}^{T} EWS(w_i)$ where $EWS(w_i)$ represents the standard EWS (e.g., autocorrelation at lag-1) calculated for the i-th sub-time series w_i.

An important step in applying the modified early warning indicators to the Poincaré sampled data involves estimating the underlying dynamic type, specifically the dominant period of the oscillation, directly from the discrete time series obtained via the Poincaré section. To estimate T, we analyze the autocorrelation function of the Poincaré time series the same way as Ref. [42]. The primary goal is to identify the lag T at which the autocorrelation function reaches its maximum (excluding the peak at lag 0). This lag T represents the estimated period T. Similar to the approach used for maps, thresholds are employed to handle transients and noisy signals, ensuring robust period estimation. Specifically, if the autocorrelation at a given lag exceeds a predefined high threshold (e.g., > 0.99999999) or the variance of the Poincaré series falls below a low threshold (e.g., < 0.001), the autocorrelation is set to one. Conversely, if the estimated period T exceeds a predefined maximum threshold, the dynamics are classified as highly complex or chaotic, and T is set to this maximum value $T_{max} = 100$ for subsequent decomposition. This step effectively determines the appropriate number of sub-time series into which the Poincaré data will be divided in the next step.

By applying this modified methodology to the Poincaré sampled data of continuous flows, we aim to overcome the limitations of both standard EWS applied directly to raw flow data and standard EWS applied to Poincaré data. This approach leverages the discrete nature of the Poincaré map while incorporating the period-based decomposition concept to enhance sensitivity to complex bifurcations, such as those occurring in period-doubling routes to chaos in

continuous neuronal models. The subsequent results section will demonstrate the effectiveness of this adaptation.

3 Results: enhanced early warning signals for bifurcation points in the HR neuron flow

This section demonstrates the application of the proposed modified EWSs, derived from the Poincaré sampled time series, to anticipate critical transitions in the HR neuron model. The bifurcation parameter r is swept from 0 to 0.05, encompassing various dynamical regimes including periodic spiking, bursting, period-doubling cascades, and chaotic behavior. The performance of the modified indicators (Autocorrelation at lag-1, Variance, Kurtosis, Skewness) is evaluated against the limitations observed when standard EWS are applied directly to the Poincaré section (as shown in Sect. 2.2.2).

3.1 Performance of modified autocorrelation (ACC)

Figure 4 presents the outcome of applying the modified ACC to the Poincaré time series (local maxima of z) from the HR model. The bifurcation diagram of maximum z values is shown in blue, while the modified ACC values are plotted in red.

In contrast to the flat or erratic behavior of standard ACC on the Poincaré signal (Fig. 3), the modified ACC exhibits distinct trends correlated with the system's approach to critical transitions. Notably, prior to period-doubling bifurcations, a discernible increase in the absolute value of the modified ACC value is observed, as the system slows down. This increasing trend is visible not only before simple periodic transitions but also ahead of transitions into more complex dynamics, including the onset of chaotic regimes via period-doubling cascades. The modified ACC effectively captures the signature of critical slowing down associated with these intricate bifurcations when analyzed through the period-decomposed components of the Poincaré map.

For example, consider the period-doubling bifurcation near $r \approx 0.037$. The modified autocorrelation (ACC) begins to rise significantly at parameter values well before the bifurcation point (e.g., from $r \approx 0.03$ onward), providing a clear anticipatory signal. Conversely, as r increases beyond the bifurcation and the system moves deeper into a new dynamical regime, the ACC decreases, reflecting reduced critical slowing down. This behavior enables a qualitative estimation of "lead time" in parameter space: the earlier the indicator deviates from baseline, the farther in advance the transition can be anticipated.

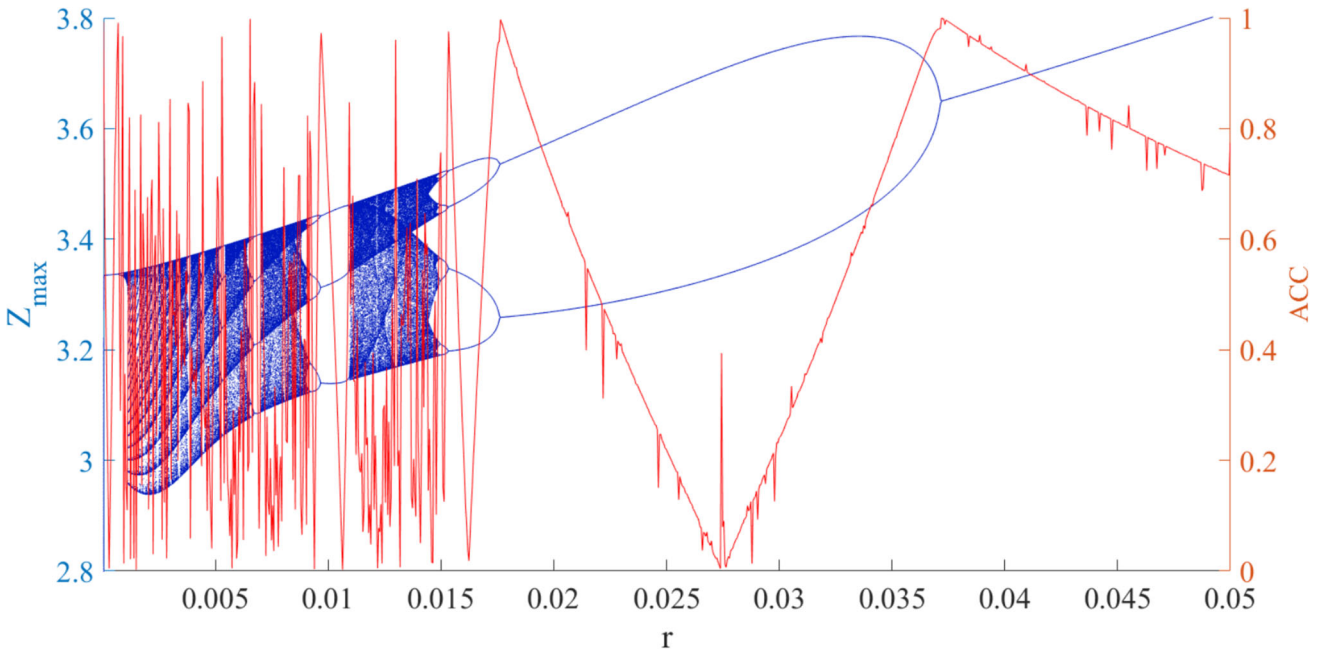


Fig. 4 Performance of the modified autocorrelation at lag-1 (ACC) applied to the Poincaré sampled signal from the HR model; the modified ACC values (red), calculated using the proposed period-based decomposition method, are plotted against the bifurcation diagram of maximum z values (blue). The modified ACC (red) exhibits clear increasing trends prior to multiple bifurcation points—including period-doubling bifurcations and transitions into or out of chaos—showing that period-aware decomposition restores sensitivity to critical slowing down in continuous chaotic systems

Similar anticipatory trends are captured for other transition types in the HR model—including intermittency and boundary crises. While our current analysis is conducted in parameter space (rather than real time), the monotonic and consistent trends of the modified indicator before each class of bifurcation effectively encode predictive lead information.

3.2 Performance of modified variance

Figure 5 illustrates the application of the modified variance calculation to the decomposed Poincaré time series. Again, the bifurcation diagram (maximum z values, blue) is juxtaposed with the modified variance (red). The modified variance shows a marked increase in the vicinity of tipping points. As the HR system approaches bifurcations—whether transitioning between different periodic states, entering a period-doubling sequence, or approaching chaotic dynamics—the modified variance rises, however there are some perturbations in the increasing and decreasing trend of the variance. This behavior aligns with the expectation that fluctuations should amplify as the system's resilience weakens near a critical transition. The rise in modified variance is particularly evident before the cascade of period-doubling bifurcations leading to chaos, demonstrating its sensitivity to the buildup of dynamical instability in the flow system when the analysis is properly adapted to the oscillatory nature of the data.

3.3 Performance of modified kurtosis

The results for the modified kurtosis are displayed in Fig. 6. The bifurcation diagram (blue) is shown alongside the modified kurtosis values (red). The modified kurtosis exhibits informative changes near bifurcation points. Often, a noticeable decrease in kurtosis is observed as the system approaches bifurcations. This trend suggests a shift towards a more Gaussian-like distribution of the states within the decomposed components, reflecting the changing nature of fluctuations near criticality. This behavior is consistent across different types of transitions observed in the HR model, including entry into chaotic regions and transitions between periodic windows. The modified kurtosis thus provides a complementary signal to variance, capturing changes in the shape of the state distribution associated with impending transitions in the continuous flow.

3.4 Performance of modified skewness

Figure 7 shows the outcome for the modified skewness applied to the HR model's Poincaré data. The bifurcation diagram (maximum z values, blue) is plotted with the

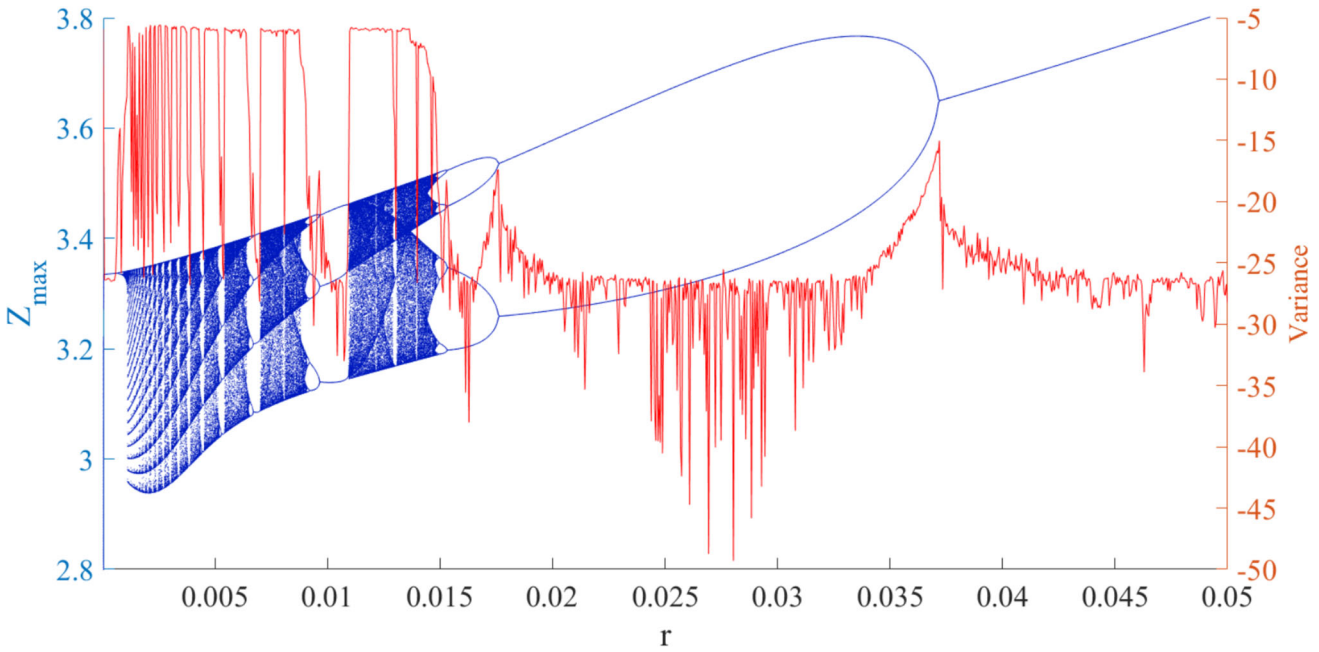


Fig. 5 Performance of the modified variance applied to the Poincaré sampled signal from the HR model; the modified variance values (red), calculated using the proposed method, are plotted against the bifurcation diagram of maximum z values (blue). The modified variance (red) increases as the system approaches bifurcation points, particularly before period-doubling cascades leading to chaos, confirming that variance—when properly adapted—can serve as a reliable early warning signal for complex transitions in continuous flows

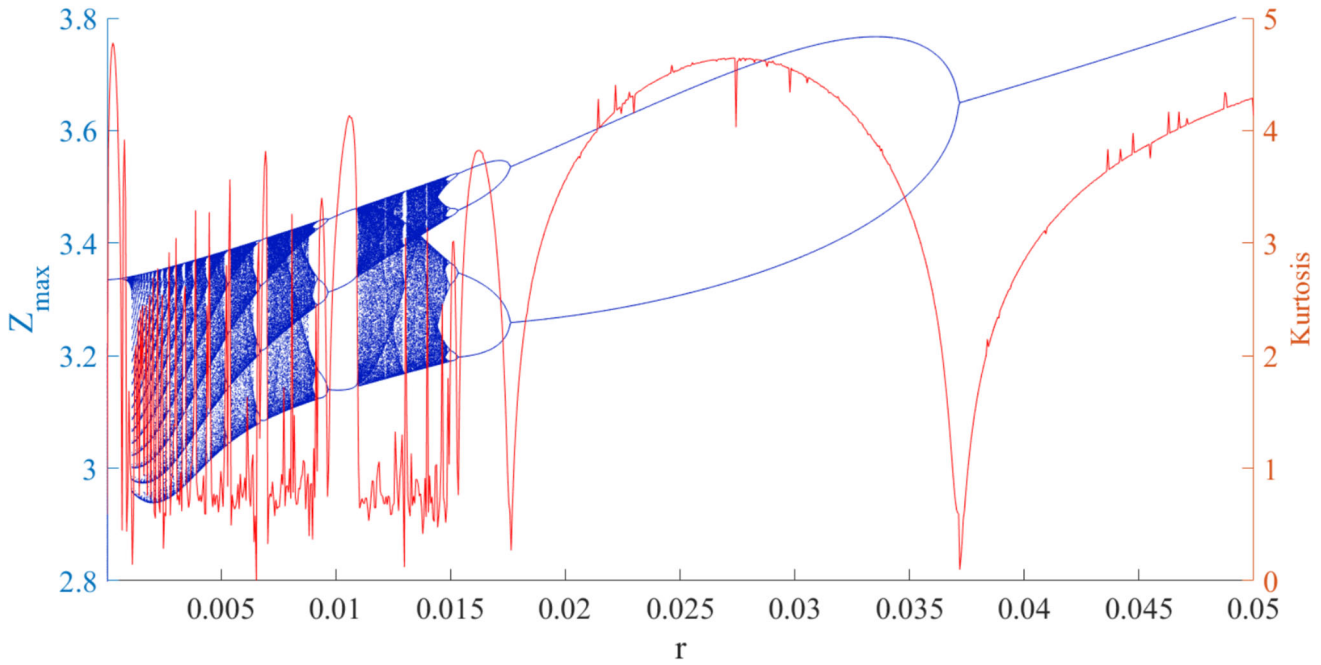


Fig. 6 Performance of the modified kurtosis applied to the Poincaré sampled signal from the HR model; the logarithm of modified kurtosis values (red), calculated using the proposed method, are plotted against the bifurcation diagram of maximum z values (blue). The modified kurtosis (red) shows systematic decreases near bifurcation points; this provides complementary evidence of critical slowing down and enhances the robustness of multi-indicator early warning frameworks

logarithm of absolute value of modified skewness (red). The modified skewness represents an exception in our method, where the computed values for each of the T subtime series are not averaged; instead, their minimum value is reported as the final index. The modified skewness demonstrates characteristic variations near critical

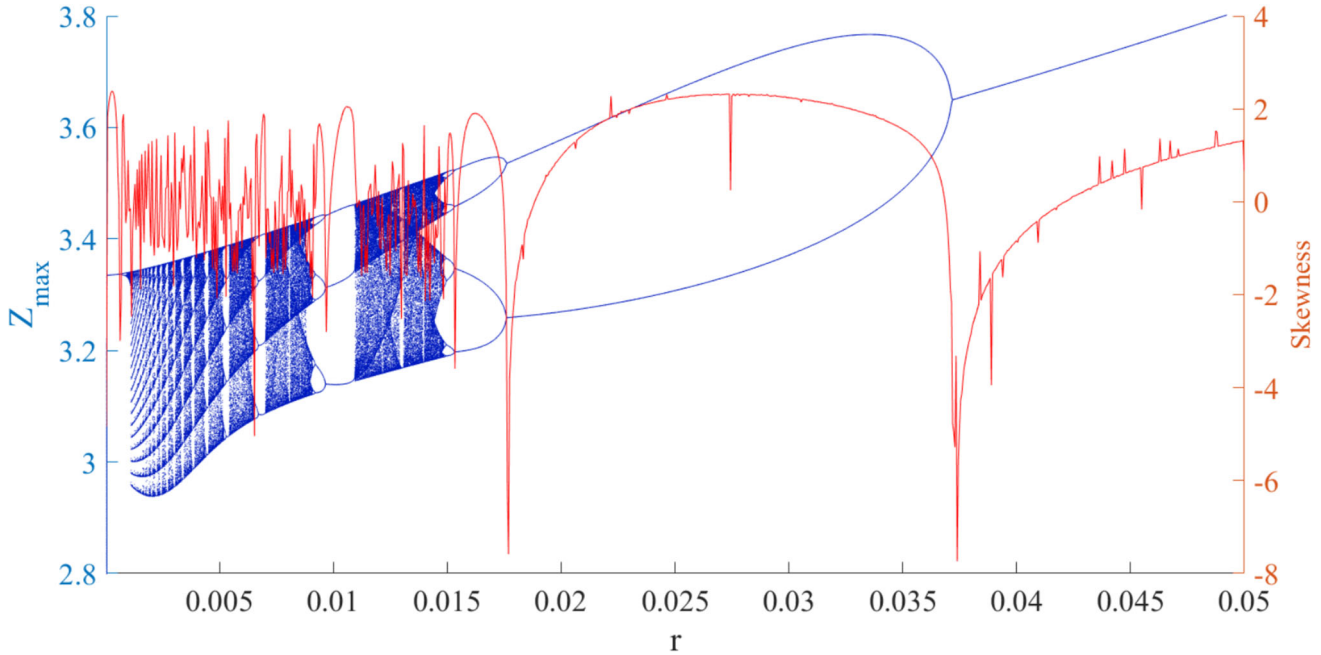


Fig. 7 Performance of the modified skewness applied to the Poincaré sampled signal from the HR model; the logarithm of the absolute value of modified skewness (red), calculated using the proposed method, is plotted against the bifurcation diagram of maximum z values (blue). The modified skewness (red) generally decreases in magnitude near bifurcations—a distinctive signature that further validates the sensitivity of the proposed method to impending critical transitions

transitions. In many cases, the absolute value of the modified skewness increases as bifurcation points are approached. This indicates a shrinking asymmetry in the system's response or state distribution on the timescale captured by the Poincaré decomposition. This decreasing trend is observable for various transitions, including period-doubling events and the boundaries of chaotic or periodic window regions.

Collectively, Figs. 4 through 7 illustrate that the proposed modification of early warning signals, when applied to the Poincaré section of the HR flow and subsequently decomposed according to the estimated period, yields indicators capable of anticipating a wide range of bifurcation points. These modified EWS overcome the significant limitations of standard indicators applied directly to the Poincaré sampled data (Sect. 2.2.2), where trends were largely absent or uninformative.

The key finding is that by focusing the EWS calculation on the period-specific components of the oscillatory flow dynamics (via Poincaré sampling and subsequent decomposition), the indicators regain their sensitivity to critical slowing down, even for complex transitions like period-doubling cascades and crises. This adaptation successfully bridges the gap between the effective method developed for discrete maps and the challenges posed by continuous dynamical systems, providing a robust framework for anticipating bifurcation points in chaotic flow-based neuron models.

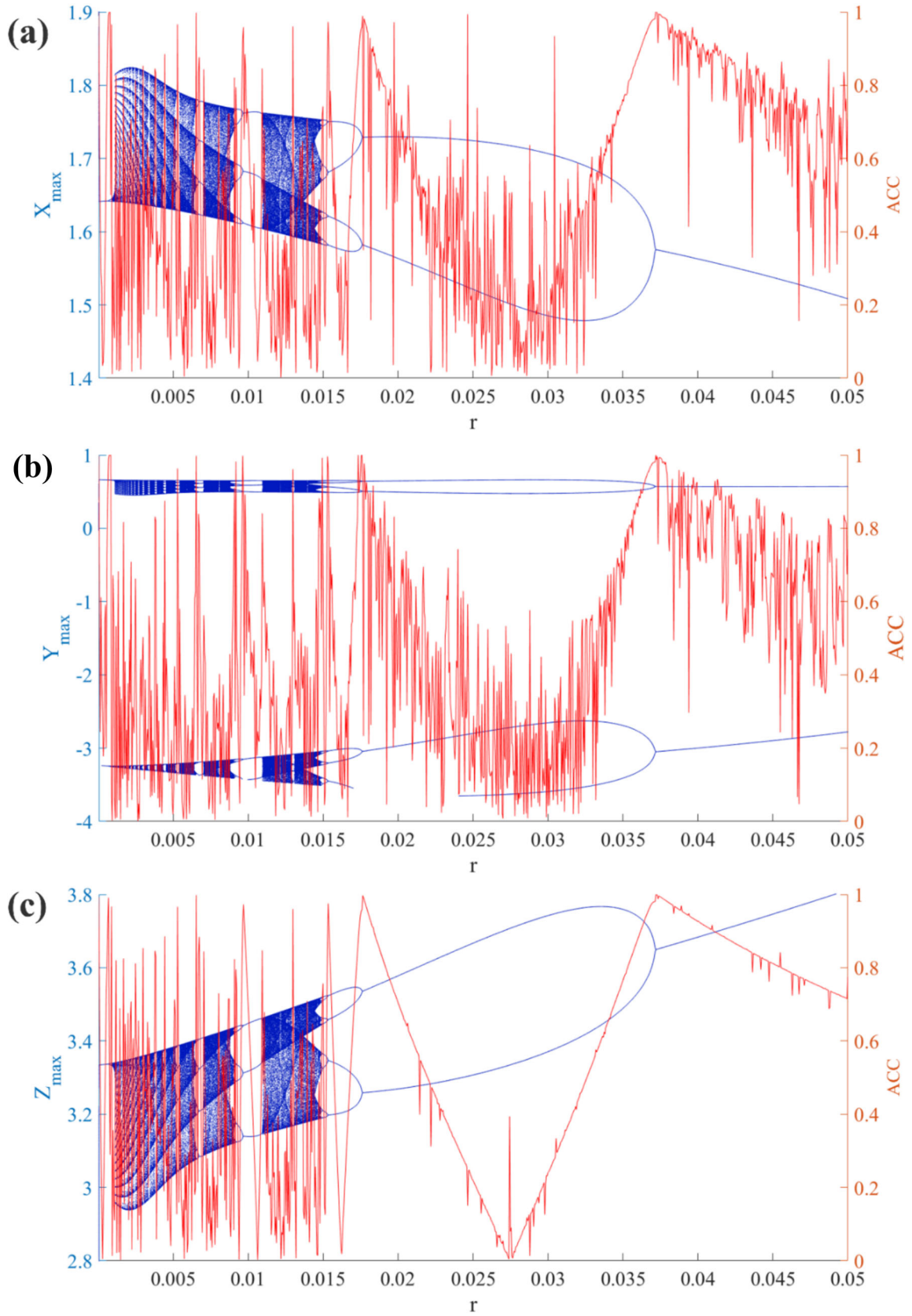
4 Discussion

The application of the proposed early warning methodology to continuous dynamical systems necessitates careful consideration of which system variables are most suitable for analysis. To investigate the robustness and potential limitations of our approach, we focus on the modified autocorrelation coefficient (ACC) and examine how the choice of observed variable affects the prediction performance. While our initial results (Sect. 3) utilized the z variable, it is crucial to understand the implications of selecting alternative state variables (x or y) from the Hindmarsh-Rose neuron model. Figure 8 presents a comparative analysis of the modified ACC computed for all three state variables (x, y, and z) of the HR model, shown in parts (a), (b), and (c) respectively. This comparison reveals several important insights regarding variable selection for early warning signal analysis in flow systems. First, a fundamental observation is that the perceived periodicity of the system dynamics can vary depending on the chosen observation variable. For instance, at $r = 0.05$, the neuron exhibits period-1 behavior when observed through the x and z variables, yet appears to display period-2 dynamics when monitored via the y variable. This phenomenon underscores the importance of understanding the relationship between the underlying flow structure and the measurement perspective, as different variables may encode the system's periodicity with varying degrees of clarity. Second, a clear distinction emerges in the quality of the computed modified ACC signals. The indicators derived from the x and y variables exhibit significantly more noise and oscillatory behavior compared to those obtained from the z variable. This increased variability in the x and y-based indicators can potentially obscure the predictive signals and reduce the reliability of the early warning system. In contrast, the modified ACC computed from the z variable demonstrates a smoother profile, which facilitates more robust interpretation of the approaching critical transitions. Despite these differences in signal quality, the fundamental predictive characteristic – the rising trend of modified ACC before bifurcation points – remains consistent across all three variables. This consistency suggests that the core mechanism of the proposed method is robust, though the clarity of its expression depends significantly on the choice of observed variable.

To elucidate the source of these differences, we examined the intermediate steps of our proposed methodology. Figure 9 presents the estimated periods computed from the time series of x, y, and z variables. Notably, the period estimation algorithm performs correctly across all three variables, accurately identifying the transitions between different dynamical regimes at the same critical parameter values of r. This finding indicates that the increased noise observed in the modified ACC for x and y variables is not attributable to failures in the period estimation step, but rather originates from subsequent stages of the analysis.

To further investigate the origin of these differences, we analyzed the underlying dynamics of the HR model. The analysis reveals a key characteristic of the HR model: the z variable evolves on a significantly slower timescale compared to the x and y variables. Additionally, the z variable has a much smaller amplitude variation. The slower dynamics allow for more precise measurement of the system's recovery timescales near bifurcation points. The reduced amplitude variations in z also contribute to the cleaner, less noisy appearance of the modified ACC computed from this variable. These properties make the z variable particularly well-suited for detecting critical slowing down (CSD).

Therefore, for optimal performance of the proposed early warning method, it is advisable to select the slowest variable of the system for analysis, as it provides the most sensitive measure of dynamical slowing. However, it is important to acknowledge that even with the optimal variable selection, some irregular spikes occasionally appear in the computed ACC values. These anomalies


(a)
1.9
1.8
1.7
1.6
1.5
1.4
X_{max}
1
0.8
0.6
0.4
0.2
0
ACC
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
r
(b)
1
0
-1
-2
-3
-4
Y_{max}
ACC
r
(c)
3.8
3.6
3.4
3.2
3
2.8
Z_{max}
ACC
r

◀**Fig. 8** Comparison of the modified autocorrelation at lag-1 (ACC) computed using different state variables of the HR model; The modified ACC is calculated for the a) x variable; b) y variable; c) z variable. While all variables exhibit rising ACC trends before bifurcations—confirming the robustness of the core method—the signal from the slow variable z is markedly smoother and less noisy than those from the fast variables x and y, underscoring the importance of selecting slow dynamical observables for reliable early warning detection

likely represent numerical artifacts arising from the computational implementation of the CSD indicators. While these rare occurrences do not fundamentally compromise the method's predictive capability, they highlight the need for careful implementation and potentially additional smoothing or validation steps in practical applications.

In conclusion, the choice of observation variable significantly impacts the quality and reliability of early warning signals derived from the proposed methodology. For the Hindmarsh-Rose model, the z variable emerges as the optimal choice due to its slow dynamics and reduced variability, leading to cleaner and more interpretable modified early warning indicators. This finding suggests that in practical applications of the method to other continuous dynamical systems, careful consideration should be given to selecting the most appropriate observable variable, ideally one that represents the system's slow dynamics.

5 Conclusion

This study successfully addressed the critical challenge of predicting bifurcation points in continuous dynamical systems, specifically focusing on chaotic flow-based neuron models like the Hindmarsh-Rose (HR) model. Building upon previous work on discrete maps, the period-based decomposition methodology was adapted and extended for application to continuous flows. The primary obstacle—applying the method directly to high-resolution flow data—was overcome by first sampling the continuous signal using a Poincaré section (specifically, the local maxima of the z variable). This step transformed the continuous dynamics into a discrete map, making analysis computationally feasible.

However, applying standard early warning signals (EWS) like autocorrelation, variance, kurtosis, and skewness directly to this Poincaré sampled signal still proved insufficient for reliably predicting complex transitions such as period-doubling cascades. Therefore, a crucial modification was introduced: estimating the dominant period of oscillation directly from the Poincaré time series and then decomposing this discrete signal into sub-time series based on the estimated period. Standard EWS were then calculated for each sub-series, and their average was taken as the modified indicator.

The results demonstrated that this proposed method significantly enhanced the performance of EWS for continuous systems. When applied to the HR neuron model across a range of bifurcation parameters ($r \in [0, 0.05]$), the modified indicators (Modified ACC, Variance, Kurtosis, Skewness) exhibited clear and consistent trends as the system approached various bifurcation points, including transitions between periodic states and the onset of chaotic dynamics via period-doubling routes. This was a marked improvement over both standard EWS applied directly to raw continuous data (which showed no trends) and standard EWS applied to Poincaré sampled data (which showed limited trends only for simple periodic behavior).

Furthermore, the analysis highlighted the importance of variable selection. Among the HR model's variables (x, y, z), the slow variable z was found to be the most suitable for generating reliable early warning signals, due to its reduced noise and clearer representation of the system's underlying slow dynamics. While some numerical artifacts were observed, the overall robustness of the modified indicators validated the effectiveness of this adaptation.

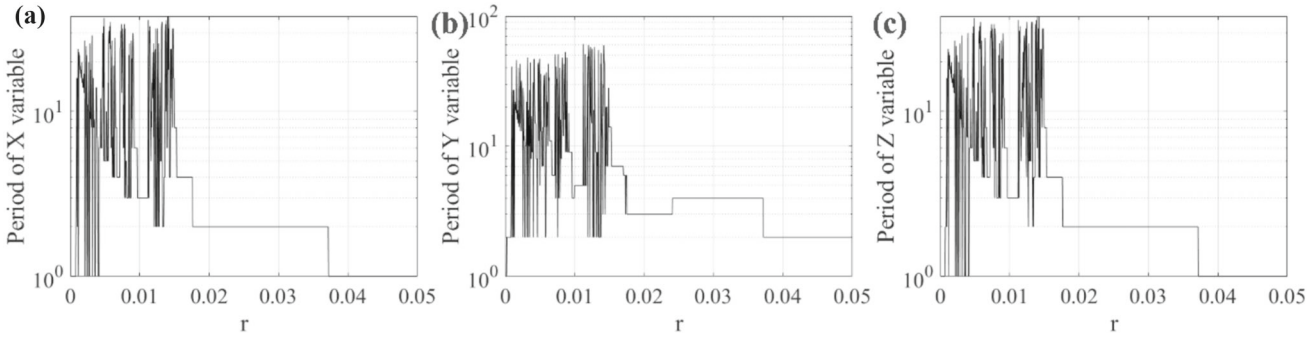


Fig. 9 Estimated dominant period of oscillation computed from the Poincaré time series derived from different state variables (x, y, z) of the HR model; the estimated periods show similar transitions at the same critical parameter values of r, confirming that the differences in modified EWS quality (Fig. 8) arise from subsequent analysis steps rather than errors in period estimation itself

In conclusion, this work bridged the gap between effective EWS methodologies for discrete maps and the complexities of continuous flows. The proposed modification provided a robust framework for anticipating critical transitions in chaotic flow-based systems, offering significant potential for applications in understanding and predicting transitions in real-world continuous dynamical systems, particularly in neuroscience.

Authors contribution S.R. performed the formal analysis and investigation, and wrote the original draft of the manuscript. F.N. contributed to the investigation and validation, and co-wrote the original draft. K.R. developed the methodology, implemented the software, and reviewed and edited the manuscript. S.J. supervised the validation process and provided critical review and editing of the manuscript. M.P. conceived the original concept, provided overall supervision, and contributed to manuscript review and editing.

Funding M.P. was supported by the Slovenian Research and Innovation Agency (Javna agencija za znanstvenoraziskovalno in inovacijsko dejavnost Republike Slovenije) (Grant No. P1-0403).

Data availability Data generated during the current study will be made available on reasonable request at the following link: https://drive.google.com/file/d/1kn_QSTsYh5WdEK_xy4VNMoZB26UPF5PK/view?usp=sharing

Declarations

Conflict of Interest The authors declare no competing interests.

References

1. Scheffer, M., Bascompte, J., Brock, W.A., Brovkin, V., Carpenter, S.R., Dakos, V., et al.: Early-warning signals for critical transitions. Nature **461**, 53 (2009)
2. Delecroix, C., van Nes, E.H., Scheffer, M., van de Leemput, I.A.: Monitoring resilience in bursts. Proc. Natl. Acad. Sci. U.S.A. **121**, e2407148121 (2024)
3. Scheffer, M., Carpenter, S.R., Lenton, T.M., Bascompte, J., Brock, W., Dakos, V., et al.: Anticipating critical transitions. Science **338**, 344–348 (2012)
4. Panahi, S., Lai, Y.-C.: Global phase-space approach to rate-induced tipping: a brief review. Chaos **35**, 043139 (2025)
5. Arumugam, R., Guichard, F., Lutscher, F.: Early warning indicators capture catastrophic transitions driven by explicit rates of environmental change. Ecology **105**, e4240 (2024)
6. Song, S., Li, H.: Early warning signals for stock market crashes: empirical and analytical insights utilizing nonlinear methods. EPJ Data Sci. **13**, 16 (2024)
7. Mehrabbeik, M., Ramamoorthy, R., Rajagopal, K., Nazarimehr, F., Jafari, S., Hussain, I.: Critical slowing down indicators in synchronous period-doubling for salamander flicker vision. Eur. Phys. J. Spec. Top. **230**, 3291–3298 (2021)
8. Nazarimehr, F., Golpayegani, S.M.R.H., Hatef, B.: Does the onset of epileptic seizure start from a bifurcation point? Eur. Phys. J. Spec. Top. **227**, 697–705 (2018)
9. Maturana, M.I., Meisel, C., Dell, K., Karoly, P.J., D'Souza, W., Grayden, D.B., et al.: Critical slowing down as a biomarker for seizure susceptibility. Nat. Commun. **11**, 2172 (2020)
10. Panahi, S., Kong, L.-W., Moradi, M., Zhai, Z.-M., Glaz, B., Haile, M., et al.: Machine learning prediction of tipping in complex dynamical systems. Phys. Rev. Res. **6**, 043194 (2024)
11. Tonge, N.A., Miller, J.P., Kharasch, E.D., Lenze, E.J., Rodebaugh, T.L.: An investigation of the potential clinical utility of critical slowing down as an early warning sign for recurrence of depression. J. Behav. Ther. Exp. Psychiatry **82**, 101922 (2024)
12. Meisel, C., Klaus, A., Kuehn, C., Plenz, D.: Critical slowing down governs the transition to neuron spiking. PLoS Comput. Biol. **11**, e1004097 (2015)
13. Shadizadeh, S.M., Nazarimehr, F., Jafari, S., Rajagopal, K.: Investigating different synaptic connections of the Chay neuron model. Phys. A Stat. Mech. Appl. **607**, 128242 (2022)
14. Xu, Q., Wang, K., Chen, M., Parastesh, F., Wang, N.: Bursting and spiking activities in a Wilson neuron circuit with memristive sodium and potassium ion channels. Chaos Solitons Fractals **181**, 114654 (2024)
15. Yao, Z., Sun, K., He, S.: Firing patterns in a fractional-order FithzHugh–Nagumo neuron model. Nonlinear Dyn. **110**, 1807–1822 (2022)
16. Ray, A., Ghosh, D.: Another new chaotic system: bifurcation and chaos control. Int. J. Bifurc. Chaos **30**, 2050161 (2020)
17. Tian, X., Wu, F., Ma, J.: Dynamics and propagation of memristive neuron models with electrical fields. Chaos Solitons Fractals **199**, 116828 (2025)
18. Zhu, Z., Dong, J., Yu, Y., Chen, Q., Ma, J.: Dynamics of synchronous encoding in vanadium dioxide (VO2) memristive neurons. Sci. China Technol. Sci. **68**, 1920406 (2025)
19. Wang, B., Ren, G., Ma, J., Guo, Y.: A memristive neuron with double capacitive variables coupled by Josephson junction. Chaos Solitons Fractals **198**, 116630 (2025)
20. Majhi, S., Ghosh, S., Pal, P.K., Pal, S., Pal, T.K., Ghosh, D., et al.: Patterns of neuronal synchrony in higher-order networks. Phys. Life Rev. **52**, 144–170 (2025)
21. Pal, P.K., Anwar, M.S., Perc, M., Ghosh, D.: Global synchronization in generalized multilayer higher-order networks. Phys. Rev. Res. **6**, 033003 (2024)
22. Yang, F., Guo, Q., Ren, G., Ma, J.: Wave propagation in a light-temperature neural network under adaptive local energy balance. J. Biol. Phys. **50**, 271–290 (2024)
23. Z. Li, Z. Li, S. He, H. Wang, and X. Wu, "Chimera states and disordered diffusion in the coupled neural networks utilizing a simplified memristive neuron," *The European Physical Journal Special Topics,* pp. 1–13, 2025.
24. Huang, W., Wu, Y., Ding, Q., Jia, Y., Xie, Y., Hu, Y.: Synchronization behavior of memristive FitzHugh-Nagumo neurons in time-varying networks under external stimuli. Chaos Solitons Fractals **192**, 116001 (2025)
25. Xie, Y., Li, X., Wang, X., Ye, Z., Hu, X., Hu, Y., et al.: Synchronization of complex networks with synapse regulated by energy difference. Nonlinear Dyn. **113**, 12099–12115 (2025)
26. Vignesh, D., He, S., Banerjee, S.: A review on the complexities of brain activity: insights from nonlinear dynamics in neuroscience. Nonlinear Dyn. **113**, 4531–4552 (2025)
27. Ma, J.: Biological neurons to neural circuit, review from physical perspective. Nonlinear Dyn. **113**, 25365–25387 (2025)
28. Ma, J., Yang, Z.-Q., Yang, L.-J., Tang, J.: A physical view of computational neurodynamics. J. Zhejiang Univ. Sci. A **20**, 639–659 (2019)
29. Ding, Q., Wu, Y., Huang, W., Hu, Y., Xie, Y., Jia, Y.: Adaptive electric shocks control and elimination of spiral waves using dynamic learning based techniques. Nonlinear Dyn. **113**, 10425–10443 (2025)

30. Yu, D., Li, X., Wang, X., Huang, W., Hu, X., Jia, Y.: Community modularity structure promotes the evolution of phase clusters and chimeralike states. Phys. Rev. E **111**, 034311 (2025)
31. Ghosh, R., Anwar, M.S., Shrimali, M.D., Ghosh, D.: Lag-mediated control of explosive synchronization transitions in adaptive multilayer networks with higher-order interactions. Chaos (2025). https://doi.org/10.1063/5.0289362
32. Lodi, M., Panahi, S., Sorrentino, F., Torcini, A., Storace, M.: Patterns of synchronized clusters in adaptive networks. Commun. Phys. **7**, 198 (2024)
33. Nazerian, A., Panahi, S., Leifer, I., Phillips, D., Makse, H.A., Sorrentino, F.: Matryoshka and disjoint cluster synchronization of networks. Chaos (2022). https://doi.org/10.1063/5.0076412
34. Hindmarsh, J., Rose, R.: A model of neuronal bursting using three coupled first order differential equations. Proc. R. Soc. Lond. B Biol. Sci. **221**, 87–102 (1984)
35. Vivekanandhan, G., Hamarash, I.I., Ali Ali, A.M., He, S., Sun, K.: Firing patterns of Izhikevich neuron model under electric field and its synchronization patterns. Eur. Phys. J. Spec. Top. **231**, 4017–4023 (2022)
36. Ma, J.: Biophysical neurons, energy, and synapse controllability: a review. J. Zhejiang Univ. Sci. A **24**, 109–129 (2023)
37. Chen, L., Bao, H., Zhang, X., Zhang, Y., Bao, B.: DC-bias induced chaotic dynamics and periodic bursting in Chua's diode-based FitzHugh-Nagumo circuit. Chaos Solitons Fractals **199**, 116739 (2025)
38. Yu, X., Bao, H., Xu, Q., Chen, M., Bao, B.: Deep brain stimulation and lag synchronization in a memristive two-neuron network. Neural Netw. **180**, 106728 (2024)
39. Klinshov, V.V., Kovalchuk, A.V., Soloviev, I.A., Maslennikov, O.V., Franović, I., Perc, M.: Extending dynamic memory of spiking neuron networks. Chaos Solitons Fractals **182**, 114850 (2024)
40. Nazarimehr, F., Jafari, S., Perc, M., Sprott, J.C.: Critical slowing down indicators. Europhys. Lett. **132**, 18001 (2020)
41. Dakos, V., Carpenter, S.R., Brock, W.A., Ellison, A.M., Guttal, V., Ives, A.R., et al.: Methods for detecting early warnings of critical transitions in time series illustrated using simulated ecological data. PLoS ONE **7**, e41010 (2012)
42. Nazarimehr, F., Jafari, S., Hashemi Golpayegani, S.M.R., Perc, M., Sprott, J.C.: Predicting tipping points of dynamical systems during a period-doubling route to chaos. Chaos **28**, 073102 (2018)
43. Chen, B., Liu, Y., Bao, H., Zhang, X., Bao, B.: Coexisting and bursting oscillations in a second-order RC-oscillator-based piecewise linear neuron circuit. Nonlinear Dyn. **113**, 17141–17159 (2025)
44. Abuhassan, K., Coyle, D., Maguire, L.P.: Investigating the neural correlates of pathological cortical networks in Alzheimer's disease using heterogeneous neuronal models. IEEE Trans. Biomed. Eng. **59**, 890–896 (2011)
45. R. C. Hilborn, *Chaos and nonlinear dynamics: an introduction for scientists and engineers*: oxford university press Oxford, 2000.